# BAYESIAN INFERENCES FOR THE BIRNBAUM-SAUNDERS SPECIAL-CASE DISTRIBUTION

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- *ABSTRACT: In this paper, we discuss the estimation of the Birnbaum-Saunders Special-Case (BS-SC) distribution through the Bayesian approach considering its parameters independents, assuming gamma priors for both of them. As the full posterior conditionals do not have closed forms we use the Metropolis-Hastings algorithm to generate samples from the joint posterior distribution. We present a simulation study proposing the Markov chain Monte Carlo (MCMC) method as a random number generator, considering the cases where the BS-SC distribution has symmetric and asymmetric shapes. An application related to ozone concentration is presented in this paper using the described methodology.*

- *KEYWORDS: Generalized Birnbaum-Saunders distributions; Markov chain Monte Carlo; Metropolis-Hastings algorithm; random number generator.*

## 1 Introduction

The Birnbaum-Saunders (BS) distribution was developed to study problems of vibration in commercial aircraft that caused fatigue in the materials (BIRNBAUM and SAUNDERS, 1969). The authors used their knowledge of fatigue problems to build a new family of distributions, which models materials lifetime subject to dynamic loads. Through the years, the BS distribution has been widespread in many works, such as Rieck and Nedelman (1991) created a log-linear model for the BS distribution; Achcar (1993) introduced the Bayesian approach on the estimation of the BS parameters; Villegas et al. (2011) introduced the BS mixed models for

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censored data; Balakrishnan et al. (2011) presented mixtured models based on the BS distribution; among others.

A random variable $T$ with parameters $\alpha > 0$ and $\beta > 0$, denoted by $T \sim \mathcal{BS}(\alpha, \beta)$, is defined in terms of the Gaussian distribution as follows

$$T = \beta \left[ \frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1} \right]^2, \tag{1}$$

where $Z \sim \mathcal{N}(0,1)$. Its probability density function (pdf) is given by

$$f_T(t) = \frac{t^{-3/2}(t+\beta)}{2\alpha\sqrt{2\pi\beta}} \exp\left\{ -\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right\},\ t > 0, \tag{2}$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter and median of the distribution. As $\alpha$ grows, the BS distribution becomes positively asymmetrical, whereas when $\alpha \to 0$, the distribution becomes symmetric around $\beta$.

Later, Díaz-García and Leiva (2005) proposed a new family of distributions so-called the Generalized Birnbaum-Saunders (GBS), defined in terms of elliptic distributions. Here, the assumption that $Z \sim \mathcal{N}(0,1)$ from equation (1) is relaxed for any univariate symmetric distribution, i.e.

$$T = \beta \left[ \frac{\alpha U}{2} + \sqrt{\left(\frac{\alpha U}{2}\right)^2 + 1} \right]^2,$$

where $U \sim \mathcal{S}(0,1;g)$, $g$ corresponds to the kernel of the pdf of symmetric distribution used and $\alpha$ and $\beta$ are the same as presented in equation (2). Thus, it follows that a random variable $T$ follows a GBS distribution, denoted by $T \sim \mathcal{GBS}(\alpha, \beta; g)$, if its pdf is given by

$$f_T(t) = c\frac{t^{-3/2}(t+\beta)}{2\alpha\sqrt{\beta}} g\left[\frac{1}{\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right],\ t > 0, \tag{3}$$

where $c$ is the normalization constant and $g$ corresponds to the kernel of the pdf of symmetric distribution to be used. In particular, when $U \sim N(0,1)$ we have the BS distribution. For instance, some other extensions of the BS distribution can be found in Vilca-Labra and Leiva (2006) who assumed that $U$ could follow any skew elliptical distribution; Owen (2006) developed a three-parameter BS distribution; Gómez et al. (2009) introduced the generalized slash Birnbaum-Saunders family of distributions; Castillo et al. (2009) proposed a new extension based on the epsilon-skew symmetric distributions; Guiraud et al. (2009) and Leiva et al. (2012) introduced a non-centrality parameter to the BS and BS-$t$ distributions; among others.

Due to their properties and flexibility in modelling different types of data, the GBS distributions received wide attention in different research areas, e.g., Leiva et al. (2008) modelled the air pollutant concentration in Chile using the GBS distributions; Leiva et al. (2012) used the GBS distributions in the forestry sciences, modelling the diameter of trees; Marchant et al. (2013) utilized distributions from the GBS family on a financial dataset. Cancho et al. (2010) present the only study using Bayesian approach on a GBS distribution besides the standard BS.

One of the GBS distributions that are not explored in the literature is the Birnbaum-Saunders Special-Case (BS-SC) distribution, also proposed by Díaz-García and Leiva (2005), which has as baseline the Special-Case distribution (for further information, see Gupta and Varga, 1993). The BS-SC model has heavier tails than the classic BS distribution and so could be used in cases where there are only a few observations on the extremes of the distribution. Also, since the BS-SC has heavy tails we can compare it with the BS-$t$ distribution but, in a Bayesian approach, the BS-SC distribution is easier to fit considering that the BS-$t$ model has a degree of freedom parameter ($\nu$) which is somewhat not very easy to estimate.

In this paper we consider the Bayesian inference as a tool for parameter estimation of the BS-SC distribution. This approach was chosen since the distribution has only its first moment and therefore becomes the natural choice for the inference process, since in the classical approach some asymptotic assumptions are violated and thus the estimates are not reliable. The modelling of uncertainty on shape ($\alpha$) and scale ($\beta$) parameters, considered independent in this work, was performed by gamma prior distributions due to their parametric space. Since the full conditional posterior distributions do not have closed form, the Metropolis-Hastings (HASTINGS, 1970) was used to obtain samples of the joint posterior distribution, and hence the Bayesian estimates.

For the simulation study, we propose the generation of data from BS-SC distribution to be performed by MCMC-based algorithms (as Metropolis-Hastings), since the quantile function of this model does not have a closed form. After data generation Bayesian estimates were obtained and compared. Finally, an application, comparing the BS, BS-$t$ and BS-SC distributions, to a real dataset related to ozone concentration in New York city is presented in order to validate the inference process.

The rest of this paper is organized as follows. In Section 2, we define the BS-SC distribution, notation and structure, comparing it to the classic BS distribution. In Section 3, prior distribution and posterior analysis are described. In Section 4, we bring up the simulation study, with data generation and its estimates. In Section 5, an illustrative example based on real data is provided. Finally, Section 6 ends with some concluding remarks.

## 2 Birnbaum-Saunders Special-Case distribution

Let X be a random variable which follows a Special-Case (SC) distribution (GUPTA and VARGA, 1993), denoted by $X \sim \mathcal{SC}(\mu, \sigma)$, so its pdf is given by

$$f_X(x) = \frac{2^{\frac{1}{2}}}{\pi\sigma}\left(1 + \left[\frac{(x-\mu)^2}{\sigma^2}\right]^2\right)^{-1}, \ x \in \mathbb{R} \tag{4}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are, respectively, location and scale parameters of the distribution.

The SC model is a symmetric distribution (GUPTA and VARGA, 1993) that has heavier tails than the Gaussian distribution and hence could be an interesting competitive model to it and to the BS-$t$ distribution when there are some extreme values in the tails of the distribution. Further, the SC distribution allocates more information around its mode as we can see in Figure 1 that presents a comparison between the BS and BS-SC models for different values of $\sigma^2$.

The only moments that can be obtained for this distribution are the first and second one. For any $n \geq 3$, $\mathrm{E}(X^n)$ does not exist since they diverge. Mean and variance of the SC distribution are given respectively by

$$\mathrm{E}(X) = \mu \quad \text{and} \quad \mathrm{Var}(X) = \sigma^2,$$

which are the same of the Gaussian distribution.

An extension of the BS distribution was proposed by Díaz-García and Leiva (2005), where they presented the family of generalized Birnbaum-Saunders (GBS) distributions, which pdf is expressed in (3). One particular case of the GBS distribution is the Birnbaum-Saunders Special-Case (BS-SC) distribution that is obtained writing the pdf (4) as equation (3).

We say that a random variable $T$ follows a BS-SC distribution, denoted as $T \sim \mathcal{BS} - \mathcal{SC}(\alpha, \beta)$, if its pdf is given by

$$f_T(t) = \frac{t^{-3/2}(t+\beta)}{\pi\alpha\sqrt{2\beta}}\left[1 + \frac{1}{\alpha^4}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)^2\right]^{-1}, \ t > 0,$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters. If $T \sim \mathcal{BS} - \mathcal{SC}(\alpha, \beta)$, then $Y = aT \sim \mathcal{BS} - \mathcal{SC}(\alpha, a\beta)$ and $Y = T^{-1} \sim \mathcal{BS} - \mathcal{SC}(\alpha, \beta^{-1})$ (DÍAZ-GARCÍA and LEIVA, 2005).

According to theorem 3 from Díaz-García and Leiva (2005), the only moment that can be obtained of the BS-SC distribution is the first one, given by

$$\mathrm{E}(T) = \beta\left(1 + \frac{\alpha^2}{2}\right),$$

which is exactly equal to the first moment of the classic BS distribution (BIRNBAUM and SAUNDERS, 1969).

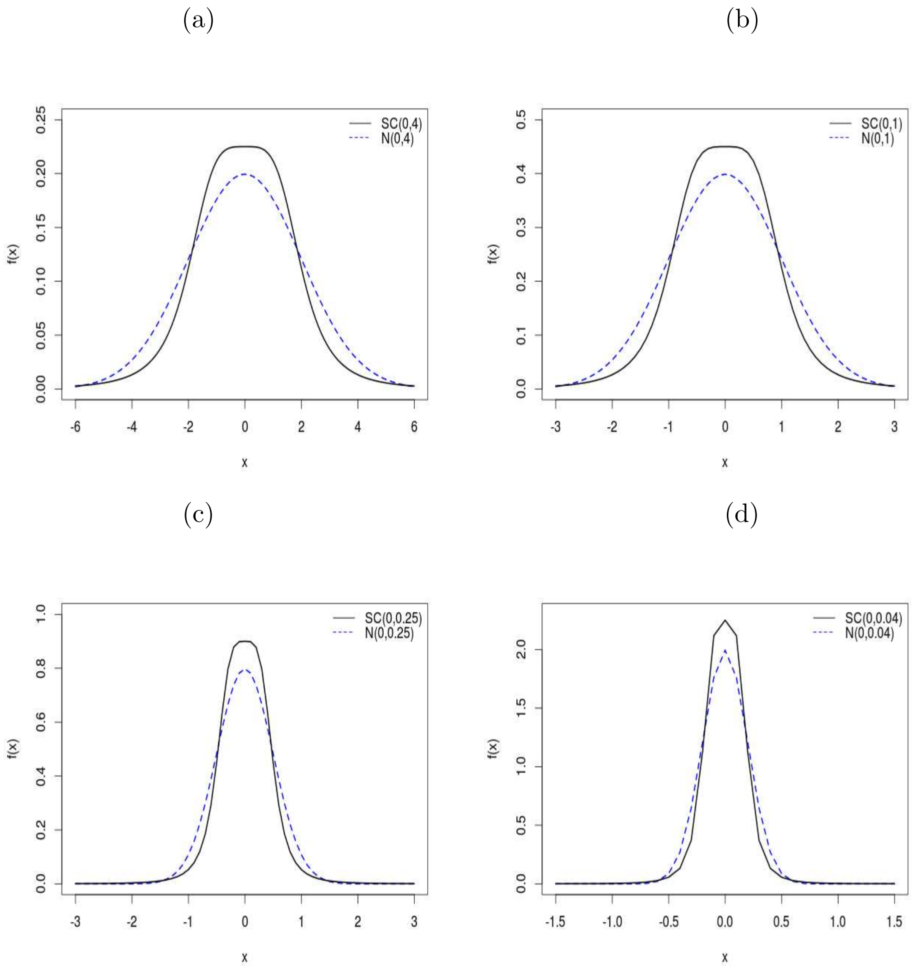


Figura 1 - Probability density functions for Gaussian and SC distributions with $\mu = 0$ and different values of $\sigma$: (a) $\sigma^2 = 4$; (b) $\sigma^2 = 1$; (c) $\sigma^2 = 0.25$; and (d) $\sigma^2 = 0.04$.

The pdf behavior of a random variable $T \sim \mathcal{BS} - \mathcal{SC}(\alpha, \beta)$ is quite similar to BS pdf (Figure 2). Graphically, the main difference between the BS-SC and BS distributions as expected from the comparison between the SC and Gaussian distribution, comes from the fact that the first one has heavier extreme tails than the second one. Also, we can observe that the BS-SC distribution allocates more observations around its mode than the BS distribution when $0 < \alpha < 1$. These two main differences make the BS-SC more attractive than the classic BS distribution in cases where some extreme values are observed on the tails of the distribution.

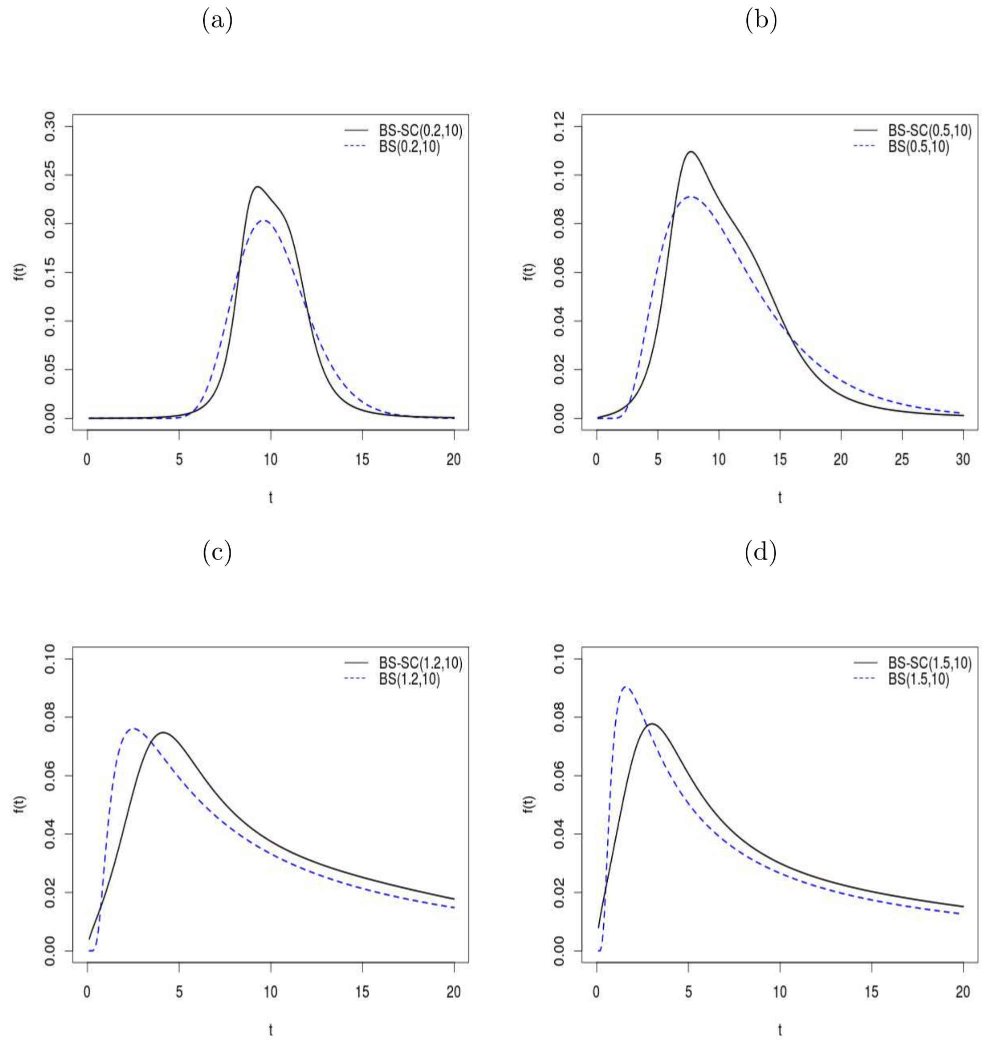


Figura 2 - Probability density functions for BS and BS-SC distributions with $\beta = 10$ and different values of $\alpha$: (a) $\alpha = 0.2$; (b) $\alpha = 0.5$; (c) $\alpha = 1.2$; and (d) $\alpha = 1.5$.

## 3 Prior distribution and posterior analysis

Let $T_1, \ldots, T_n$ be independent and identically distributed random variables, where $T_i \sim \mathcal{BS} - \mathcal{SC}(\alpha, \beta)$, $i = 1, \ldots, n$. A useful reparametrization for the classic BS distribution is $\lambda = \alpha^{-2}$ since we can take a conditionally conjugate gamma prior for $\lambda$. Here we use the same reparametrization although the conjugate property is not valid in our case. Thus, setting $\lambda = \alpha^{-2}$, the BS-SC likelihood function, without normalization constant, can be written as

$$L(\lambda, \beta|\mathcal{D}) \propto \frac{\lambda^{n/2}}{\beta^{n/2}} \frac{\prod_{i=1}^{n} t_i^{-3/2}\left(t_i + \beta\right)}{\prod_{i=1}^{n}\left[1 + \lambda^2\left(\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right)^2\right]}, \ \lambda > 0, \tag{5}$$

where $\mathcal{D}$ denotes the data.

The uncertainty of the parameters $\lambda$ and $\beta$, considered to have independent prior distributions, is described as

$$\pi(\lambda) \propto \lambda^{a-1}\mathrm{e}^{-b\lambda}, \quad \lambda > 0, \tag{6}$$

and

$$\pi(\beta) \propto \beta^{c-1}\mathrm{e}^{-d\beta}, \quad \beta > 0, \tag{7}$$

i.e., we used the gamma distribution with shape and rate hyperparameters $a$ and $b$, respectively, for $\lambda$ and gamma distribution with shape and rate hyperparamenters $c$ and $d$, respectively, for $\beta$.

Combining the information from data in equation (5), with the prior information from equations (6) and (7), we obtain the joint posterior density function of $(\lambda,\beta)$, i.e.

$$\pi(\lambda, \beta|\mathcal{D}) \propto \frac{\lambda^{\frac{n}{2}+a-1}\mathrm{e}^{-b\lambda-d\beta}}{\beta^{\frac{n}{2}-c+1}} \frac{\prod_{i=1}^{n} t_i^{-3/2}\left(t_i + \beta\right)}{\prod_{i=1}^{n}\left[1 + \lambda^2\left(\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right)^2\right]}. \tag{8}$$

Therefore, the marginal posterior distributions are easily obtained from equation (8) as follows

$$\pi(\lambda|\beta, \mathcal{D}) \propto \frac{\lambda^{\frac{n}{2}+a-1}\mathrm{e}^{-b\lambda}}{\prod_{i=1}^{n}\left[1 + \lambda^2\left(\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right)^2\right]}$$

and

$$\pi(\beta|\lambda, \mathcal{D}) \propto \beta^{c-\frac{n}{2}-1}\mathrm{e}^{-d\beta} \frac{\prod_{i=1}^{n} t_i^{-\frac{3}{2}}\left(t_i + \beta\right)}{\prod_{i=1}^{n}\left[1 + \lambda^2\left(\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right)^2\right]}.$$

We can observe that the marginal posterior distributions do not have closed form and, thus, to acquire samples from the joint posterior distribution the Metropolis-Hastings (HASTINGS, 1970) algorithm will be used.

## 4 Simulation study

Since the quantile function of the BS-SC distribution has no closed form, we propose in this paper the data generation of this model via Metropolis-Hastings algorithm. The steps to obtain the observations are described below:

- Step 1: Establish an initial value for the start of the algorithm, denoted by $y^{(0)}$;
- Step 2: $y^{(i+1)} = y^{(i)}$, where $y^{(i)}$, $i = 0, \ldots, M-1$, is the new sample of the chain;
- Step 3: Generate a new candidate $y_{\text{new}}$ from a proposal distribution $g(y)$;
- Step 4: Generate $u$ from an Uniform(0,1);
- Step 5: If $u > \frac{f(y^{(i)})}{f(y_{\text{new}})} \frac{g(y_{\text{new}})}{g(y^{(i)})}$ we should keep the observation $y^{(i)}$, otherwise $y^{(i)} = y_{\text{new}}$;
- Step 6: Repeat Steps 2 to 5 until a certain number of observations $M$ is obtained.

It is noteworthy that the acceptance rate should be maintained between 25% and 45%, considering that a low acceptance rate may indicate that the sample values are in the distribution tails, while a high acceptance rate may indicate that the values are being sampled only from regions with high probability density.

In this study we generate four different scenarios with the BS-SC model, covering cases where the shape of the distribution is near symmetrical ($\alpha = 0.2$) or very asymmetrical ($\alpha = 1.5$):

- Scenario 1: $\mathcal{BS} - \mathcal{SC}(\alpha = 0.2, \beta = 1.5)$
- Scenario 2: $\mathcal{BS} - \mathcal{SC}(\alpha = 0.2, \beta = 0.2)$
- Scenario 3: $\mathcal{BS} - \mathcal{SC}(\alpha = 1.5, \beta = 1.5)$
- Scenario 4: $\mathcal{BS} - \mathcal{SC}(\alpha = 1.5, \beta = 0.2)$

For each scenario we used five different sample sizes ($n_1 = 15$, $n_2 = 20$, $n_3 = 30$, $n_4 = 50$ and $n_5 = 100$) and generated 1,000 datasets. In computing the Bayesian estimates we ran 50,000 iterations, with a burn-in=10,000 and thin=10. For prior information we have used two different gamma priors: i) Prior 1 is a non-informative prior with hyperparameters $a = b = c = d = 0.01$; and ii) Prior 2 is an informative prior in which the hyper-parameters was chosen in such a way that the prior mean became the expected value of the corresponding parameter. All the simulation study was performed on R software (R CORE TEAM, 2013) in a HP Proliant M530e Gen8 computer.

Table 1 presents the posterior mean for both parameters, $\alpha$ and $\beta$, obtained from the Bayesian methods, as well as their Monte Carlo errors (in parentheses) for both priors.

Tabela 1 - Average estimates and the associated Monte Carlo errors for the Bayesian approach of the simulation from the BS-SC distribution with different values of $\alpha$ and $\beta$

| Empirical distribution | Number of observations | Prior 1 | | Prior 2 | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
| | 15 | 0.2144 (0.0009) | 1.5037 (0.0200) | 0.2086 (0.0009) | 1.5031 (0.0011) |
| $\alpha = 0.2$ | 20 | 0.2106 (0.0007) | 1.5038 (0.0013) | 0.2065 (0.0007) | 1.5031 (0.0009) |
| $\beta = 1.5$ | 30 | 0.2056 (0.0005) | 1.5024 (0.0007) | 0.2030 (0.0005) | 1.5022 (0.0007) |
| | 50 | 0.2036 (0.0004) | 1.5007 (0.0005) | 0.2022 (0.0003) | 1.5006 (0.0005) |
| | 100 | 0.2018 (0.0002) | 1.5006 (0.0004) | 0.2012 (0.0002) | 1.5006 (0.0004) |
| | 15 | 0.2121 (0.0011) | 0.2004 (0.0002) | 0.2064 (0.0009) | 0.2004 (0.0001) |
| $\alpha = 0.2$ | 20 | 0.2120 (0.0008) | 0.2004 (0.0001) | 0.2072 (0.0007) | 0.2004 (0.0001) |
| $\beta = 0.2$ | 30 | 0.2059 (0.0005) | 0.2009 (0.0001) | 0.2036 (0.0005) | 0.2001 (0.0001) |
| | 50 | 0.2040 (0.0004) | 0.2002 (<0.0001) | 0.2026 (0.0003) | 0.2001 (<0.0001) |
| | 100 | 0.2021 (0.0003) | 0.2001 (<0.0001) | 0.2014 (0.0002) | 0.2000 (<0.0001) |
| | 15 | 1.5591 (0.0142) | 1.5812 (0.0236) | 1.5260 (0.0204) | 1.5115 (0.0055) |
| $\alpha = 1.5$ | 20 | 1.5572 (0.0106) | 1.5675 (0.0191) | 1.5260 (0.0156) | 1.5123 (0.0050) |
| $\beta = 1.5$ | 30 | 1.5263 (0.0063) | 1.5596 (0.0122) | 1.5178 (0.0160) | 1.5109 (0.0043) |
| | 50 | 1.5201 (0.0043) | 1.5399 (0.0067) | 1.5209 (0.0063) | 1.5100 (0.0035) |
| | 100 | 1.5156 (0.0031) | 1.5256 (0.0031) | 1.5141 (0.0030) | 1.5092 (0.0026) |
| | 15 | 1.5942 (0.0117) | 0.2198 (0.0017) | 1.5348 (0.0205) | 0.2012 (0.0024) |
| $\alpha = 1.5$ | 20 | 1.5428 (0.0066) | 0.2150 (0.0011) | 1.5122 (0.0192) | 0.2011 (0.0021) |
| $\beta = 0.2$ | 30 | 1.5404 (0.0046) | 0.2102 (0.0007) | 1.5109 (0.0110) | 0.2007 (0.0020) |
| | 50 | 1.5129 (0.0059) | 0.2039 (0.0005) | 1.5096 (0.0061) | 0.2002 (0.0016) |
| | 100 | 1.5028 (0.0029) | 0.2012 (0.0003) | 1.5022 (0.0030) | 0.2005 (0.0009) |

Clearly the posterior means that were calculated are really close to the real simulated values, indicating that both simulation and inference processes are satisfactory. Further, as expected, the informative prior (Prior 2) outperformed the non-informative prior (Prior 1), especially when the distribution is asymmetrical ($\alpha = 1.5$) with a low sample size.

## 5 Application

In this section we illustrate the proposed methodology to estimate the parameters of the BS-SC distribution in a real dataset that refers to the ozone concentration in New York city in 1973. This dataset is available on `lattice` package in `R` under the name `environmental` and further details can be obtained in Bruntz et al. (1974).

Non-informative prior distributions for the parameters $\alpha$ and $\beta$ of the BS, BS-SC and BS-$t$ distributions, considered to be independent, were used to obtain the Bayesian estimates ($\lambda = 1/\alpha^2 \sim Gamma(0.01, 0.01)$ and $\beta \sim Gamma(0.01, 0.01)$). Moreover, for the BS-$t$ distribution it was considered the uniform distribution as a prior distribution for the inverse of $\nu$, i.e., $1/\nu \sim U(0.1, 0.5)$ that is somewhat informative but it was necessary in order to obtain the convergence for all three parameters of the model.

Two chains were generated for each model (Figure 3 presents the ones related to the BS-SC distribution) by Metropolis-Hastings algorithm with 50,000 iterations, where the first 10,000 were discarded as a burn-in and it was used a thin of 10 in this case. Both chains converged according to the Gelman & Rubin criterion (GELMAN and RUBIN, 1992). Furthermore, the autocorrelation of the parameters is well controlled. Therefore, according to these indications, there is no problem on the posterior statistics.

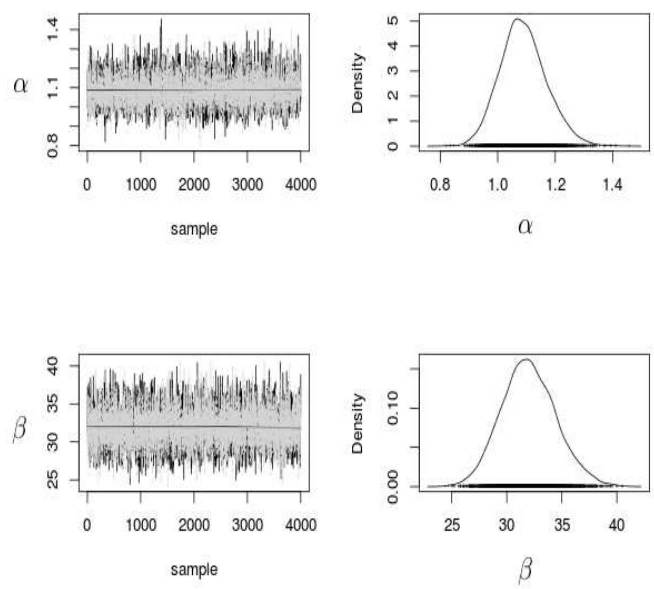


Figura 3 - History of generated chains and their densities of the parameters $\alpha$ and $\beta$ from BS-SC distribution, for the ozone concentration dataset.

Table 2 provides posterior means, standard deviations and the 95% highest posterior density (HPD) credible intervals of the parameters of the BS-SC, BS and BS-$t$ distributions. Moreover, Table 2 displays the deviance information criterion (DIC) value in order to compare these models (smaller values of DIC provide better fit, see Carlin and Louis, 2009). DIC was used since it is the most common goodness-of-fit measure in Bayesian analysis (GELMAN et al., 2013). We can observe that the parameters standard deviations for the distributions are not numerically high when compared to the posterior mean itself, excepting for $\nu$ that is actually expected. Furthermore, the HPD amplitude is not high, indicating that the parameters estimates are satisfactory (Table 2). Finally, we can say that the BS-SC distribution is the best model since it returned smaller value of DIC (2551.243). The fit of the BS-SC, BS and BS-$t$ distributions, using Bayesian approach, to the dataset in study, can be seen on Figure 4.

Tabela 2 - Posterior means, standard deviations and 95% HPD credible intervals of parameters from the BS-SC distribution of the ozone concentration dataset

| | Parameter | Estimate | Standard deviation | Lower HPD (95%) | Upper HPD (95%) | DIC |
|---|---|---|---|---|---|---|
| BS-SC | $\alpha$ | 1.0880 | 0.0808 | 0.9376 | 1.2531 | 2551.243 |
| | $\beta$ | 31.9780 | 2.4781 | 27.3799 | 37.0678 | |
| BS | $\alpha$ | 0.9994 | 0.0690 | 0.8701 | 1.1379 | 2631.518 |
| | $\beta$ | 27.9995 | 2.3373 | 23.7091 | 32.9264 | |
| BS-$t$ | $\alpha$ | 0.8235 | 0.0737 | 0.6808 | 0.9642 | 2592.069 |
| | $\beta$ | 31.0800 | 2.5860 | 26.1370 | 36.2266 | |
| | $\nu$ | 8.4160 | 7.4744 | 2.5073 | 15.1702 | |

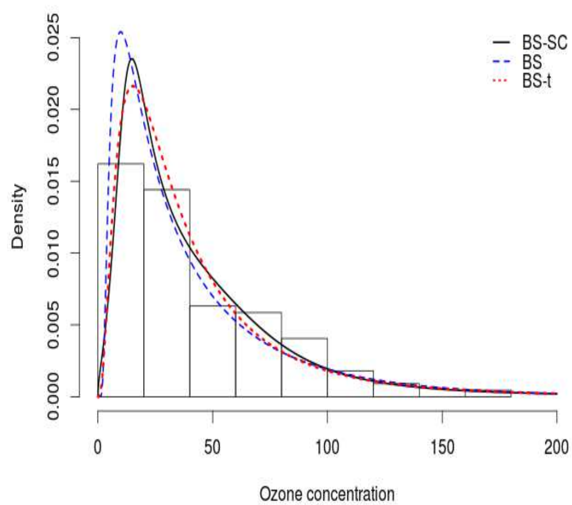


Figura 4 - Histogram of the ozone dataset and the fitted curve from BS-SC and BS distributions.

## 6 Concluding remarks

In this paper we presented the Bayesian inference as an alternative to be used in parameters estimation of the Birnbaum-Saunders Special-Case distribution since only the first moment of this distribution can be obtained, and then the frequentist approach should be avoided as some asymptotic properties are violated. We showed that there is no closed conditional posterior distributions when the gamma distribution – intuitively assumed due to the parametric spaces – with independent parameters is assumed as a prior distribution and, thus, the Metropolis-Hastings algorithm is required to generate the MCMC samples. However, as elucidated in the simulation study and in the real dataset application, the estimates for parameters $\alpha$ and $\beta$ obtained by this approach are satisfactory. Furthermore, we showed that it is possible to use the Metropolis-Hastings algorithm for the simulation of BS-SC data in an accurate way and it possibly could be used in any model. We presented one application related to the ozone concentration in New York city showing that, despite the similarity between the BS-SC distribution and the BS standard model, the BS-SC distribution fitted better according to the deviance information criterion. Finally, the Bayesian methodology applied to this work, on estimation and data simulation, and on problems involving BS-SC distribution was shown to be extremely efficient and interesting.

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- RESUMO: Neste artigo, a estimativa dos parâmetros da distribuição Birnbaum-Saunders Caso Especial (BS-SC) é realizada por meio de uma abordagem bayesiana, considerando-os independentes e utilizando a distribuição gama como priori para ambos. Uma vez que as distribuições condicionais a posteriori completas não possuem forma fechada conhecida, o algoritmo de Metropolis-Hastings foi utilizado para a obtenção de amostras da distribuição a posteriori conjunta. Um estudo de simulação é conduzido e o método de Monte Carlo via Cadeia de Markov (MCMC) é proposto como um gerador de números aleatórios da distribuição em estudo, considerando os casos em que a mesma assume a forma simétrica e assimétrica. Finalmente, uma aplicação relacionada à concentração de ozônio é apresentada neste artigo.

- PALAVRAS-CHAVE: Distribuições Birnbaum-Saunders generalizadas; Monte Carlo via cadeias de Markov; algoritmo Metropolis-Hastings; gerador de números aleatórios.

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