A DISCRETE ANALOGUE OF THE CONTINUOUS POWER LINDLEY DISTRIBUTION AND ITS APPLICATIONS

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▪ ABSTRACT: The methods to generate a discrete analogue of a continuous distribution has been widely considered in recent decades. In general, the discretization procedure comprises in transform continuous attributes into discrete attributes generating new probability distributions that could be an alternative to the traditional discrete models, such as Poisson and Binomial models, commonly used in analysis of count data. It also avoids the use of continuous in the analysis of strictly discrete data. In this paper, using the discretization method based on the survival function, it is introduced a discrete analogue of power Lindley distribution. Some mathematical properties are studied. The maximum likelihood theory is considered for estimation and asymptotic inference concerns. A simulation study is also carried out in order to evaluate some properties of the maximum likelihood estimators of the proposed model. The usefulness and accurate of the proposed model are evaluated using real datasets provided by the literature.

▪ KEYWORDS: Discretization; power Lindley distribution; Monte Carlo simulation; maximum likelihood estimators.

1 Introduction

The methods to generate a discrete analogue of a continuous distribution has been widely considered and studied in recent decades by several authors such as Good (1953), Nakagawa and Osaki (1975), Roy and Ghosh (2009) and Ghosh *et al.* (2013). In general, according to Boulle (2004), the discretization procedure

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comprises in transform continuous attributes into discrete attributes generating new probability distributions with a defined support in $\mathbb{Z}$. Basically, this process produces probability functions that could be an alternative to the traditional distributions used in count data analyze such as Poisson and Negative Binomial distributions, for example.

In the literature, there are several methods to obtain a discrete distribution from a continuous distribution: the discretization method based on the survival function (NAKAGAWA and OSAKI, 1975), the discretization method based on an infinite series (GOOD, 1953; KULASEKERA and TONKYN, 1992; KEMP, 1997; SATO *et al.*, 1999), the discretization method based on the hazard function (STEIN, 1984), the compound two-phase method (CHAKRABORTY, 2015), the discretization method based on reverse hazard function (GHOSH *et al.*, 2013), among many others.

The method of discretization by survival function was proposed by Nakagawa and Osaki (1975). This method allow us to discretize a continuous random variable from its survival function. Several properties of the survival and of the risk functions were studied by Bracquemond and Gaudoin (2003), Roy (2003), Kemp (2004), Chakraborty (2015), among many others. According to Kemp (2004), we can define an discrete analogue to continuous random variable as follows:

Definition 1.1.: *Let X a continuous random variable. If X has survival function $S_X(x)$, then the discrete random variable $Y = \lfloor X \rfloor$, where $\lfloor X \rfloor$ indicates the smallest integer part or equal to X, has PMF (probability mass function) written as:*

$$P(Y = k) = \sum_{j=0}^{1} (-1)^j S_X(k + j). \tag{1}$$

Some distributions discretized by this method introduced in the literature are: inverse Rayleigh distribution (HUSSAIN AND AHMAD, 2014), Lindley distribution (GÓMEZ-DÉNIZ and CALDERÍN-OJEDA, 2011; BAKOUCH *et al.*, 2014) , Type II generalized exponential distribution (NEKOUKHOU *et al.*, 2013), gamma distribution (CHAKRABORTY and CHAKRAVARTY, 2012), inverse Weibull distribution (JAZI *et al.*, 2010), Burr XII and Pareto distributions (KRISHNA and PUNDIR, 2009), Rayleigh distribution (ROY, 2004), among many others.

The main goal of this paper is to use Nakagawa and Osaki's discretization method to propose a discrete analogue for power Lindley distribution (GHITANY *et al.*, 2013). It is expect the proposed model to be suitable alternatives to model with good performance and accurate count and failure times datasets since not many of the known distributions (especially discrete distributions) can provide accurate models for both count and failure times data.

In the literature, the Lindley distribution was introduced by Lindley (1958) in the Bayesian context, and subsequently studies in details by Ghitany *et al.* (2008). For many years, it has been used in compound process allied with Poisson distribution (SANKARAN, 1970). A continuous random variable X is said to have

Lindley distribution if its probability density function (pdf) can be written as

$$f(x \mid \beta) = \frac{\beta^2}{1+\beta}(1+x)e^{-\beta x}, \tag{2}$$

where $x > 0$ and $\beta > 0$ is the scale parameter. The expression (2) can also be written as a mixture of two distributions which components are the exponential distribution, $f_1(x \mid \beta) = \beta e^{-\beta x}$, and the gamma distribution, $f_2(x \mid \beta) = x\beta^2 e^{-\beta x}$. The probabilities of each component are, respectively, $\frac{\beta}{1+\beta}$ and $\frac{1}{1+\beta}$.

A comprehensive discussion about the mathematical properties of the Lindley distribution such as moments, hazard function, stochastic orderings, parameter estimation, among others is also presented on the mentioned paper. The corresponding survival function is given by

$$S(x \mid \beta) = \left(1 + \frac{\beta x}{1+\beta}\right)e^{-\beta x}, \tag{3}$$

In last years, several generalizations of Lindley distributions were proposed in the literature, such as the power Lindley distribution introduced by Ghitany *et. al.* (2013), the weighted Lindley distribbution proposed by Ghitany *et. al.* (2011), the quasi-Lindley distribution introduced by Shanker (2013), the inverse Lindley distribution proposed by Sharma (2015), the transmuted Lindley distribution proposed by Merovci (2013), the inverse power Lindley proposed by Parede *et al.* (2016) and so on.

Let a random variable Y follows one-parameter Lindley distribution then $X = Y^{\frac{1}{\alpha}}$, follows a power Lindley distribution with probability density function:

$$f(x \mid \alpha, \beta) = \frac{\alpha\beta^2}{1+\beta}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \tag{4}$$

where $x > 0$ and $\alpha, \beta > 0$ are, respectively, the shape and scale parameters. The pdf in (4) has decreasing, unimodal, and decreasing-increasing-decreasing behavior. Also, note that, the Lindley distribution is a particular case of power Lindley distribution when $\alpha = 1$. Besides that, it is easy to verify that (4) may be acquired by the mixture of a Weibull distribution with shape α and scale β and the generalized gamma distribution with shape 2 and α and scale β. The corresponding survival function is given by

$$S(x \mid \alpha, \beta) = \left(1 + \frac{\beta x^\alpha}{1+\beta}\right)e^{-\beta x^\alpha}, \tag{5}$$

The contents of this paper are organized as follows: in Section 2 is presented the discrete power Lindley distribution and its mathematical properties. The estimation procedures using the method of maximum likelihood is introduced in Section 3. In Section 4 it is presented the results of a Monte Carlo simulation study to evaluate the bias and the mean squared errors of the maximum likelihood estimators. In Section 5, applications of the proposed model to real datasets are considered to illustrate its usefulness. Finally, the Section 6 close the paper with some concluding remarks.

2 The discrete power Lindley distribution

Assuming the power Lindley distribution as baseline distribution with the survival function given by (5) and using the discretization method based on the survival function, the discrete power Lindley distribution, hereafter DPL distribution, has probability function written as:

$$P(X = x \mid \alpha, \beta) = \left(1 + \frac{\beta x^{\alpha}}{\beta + 1}\right)\gamma^{x^{\alpha}} - \left(1 + \frac{\beta(x+1)^{\alpha}}{\beta + 1}\right)\gamma^{(x+1)^{\alpha}} \tag{6}$$

where $x = 0, 1, \ldots,$ $\alpha, \beta > 0$ and $\gamma = \exp(-\beta)$. The DPL distribution satisfies the log-concavity inequation (KEILSON and GERBER, 1971) and, therefore, is unimodal for all $\alpha, \beta > 0$. In Figure 1, for some values of α and β, it is illustrated the behavior of probability mass function of DPL distribution.

Theorem 2.1. *The probability mass function of DPL distribution is unimodal.*

Proof. Let a discrete random variable X such that $X \sim DPL(\alpha, \beta)$. Notice that:

$$\begin{aligned}
[P(X = x \mid \alpha, \beta)]^2 &= \gamma^{2x^{\alpha}}\left[1 + \frac{2\beta x^{\alpha}}{\beta+1} + \frac{\beta^2 x^{2\alpha}}{(\beta+1)^2}\right] \\
&- 2\gamma^{x^{\alpha}+(x+1)^{\alpha}}\left[1 + \frac{\beta(x^{\alpha} + (x+1)^{\alpha})}{\beta+1} + \frac{\beta^2 x^{\alpha}(x+1)^{\alpha}}{(\beta+1)^2}\right] \\
&+ \gamma^{2(x+1)^{\alpha}}\left[1 + \frac{2\beta(x+1)^{\alpha}}{\beta+1} + \frac{\beta^2 (x+1)^{2\alpha}}{(\beta+1)^2}\right] \\
&\geq \gamma^{(x-1)^{\alpha}+(x+1)^{\alpha}}\left[1 + \frac{\beta[(x-1)^{\alpha} + (x+1)^{\alpha}]}{\beta+1} + \frac{\beta^2 (x-1)^{\alpha}(x+1)^{\alpha}}{(\beta+1)^2}\right] \\
&- \gamma^{x^{\alpha}+(x+1)^{\alpha}}\left[1 + \frac{\beta[x^{\alpha} + (x+1)^{\alpha}]}{\beta+1} + \frac{\beta^2 x^{\alpha}(x+1)^{\alpha}}{(\beta+1)^2}\right] \\
&- \gamma^{(x-1)^{\alpha}+(x+2)^{\alpha}}\left[1 + \frac{\beta[(x-1)^{\alpha} + (x+2)^{\alpha}]}{\beta+1} + \frac{\beta^2 (x-1)^{\alpha}(x+2)^{\alpha}}{(\beta+1)^2}\right] \\
&+ \gamma^{x^{\alpha}+(x+2)^{\alpha}}\left[1 + \frac{\beta[x^{\alpha} + (x+2)^{\alpha}]}{\beta+1} + \frac{\beta^2 x^{\alpha}(x+2)^{\alpha}}{(\beta+1)^2}\right].
\end{aligned}$$

The right side of inequality above is the same as $P(X = x-1 \mid \alpha, \beta)P(X = x+1 \mid \alpha, \beta)$. Therefore:

$$[P(X = x \mid \alpha, \beta)]^2 \geq P(X = x-1 \mid \alpha, \beta)P(X = x+1 \mid \alpha, \beta).$$

In this way, the probability mass function of DPL distribution satisfies the log-concavity inequation and the result follows from Theorem 3 from Keilson and Gerber (1971). □

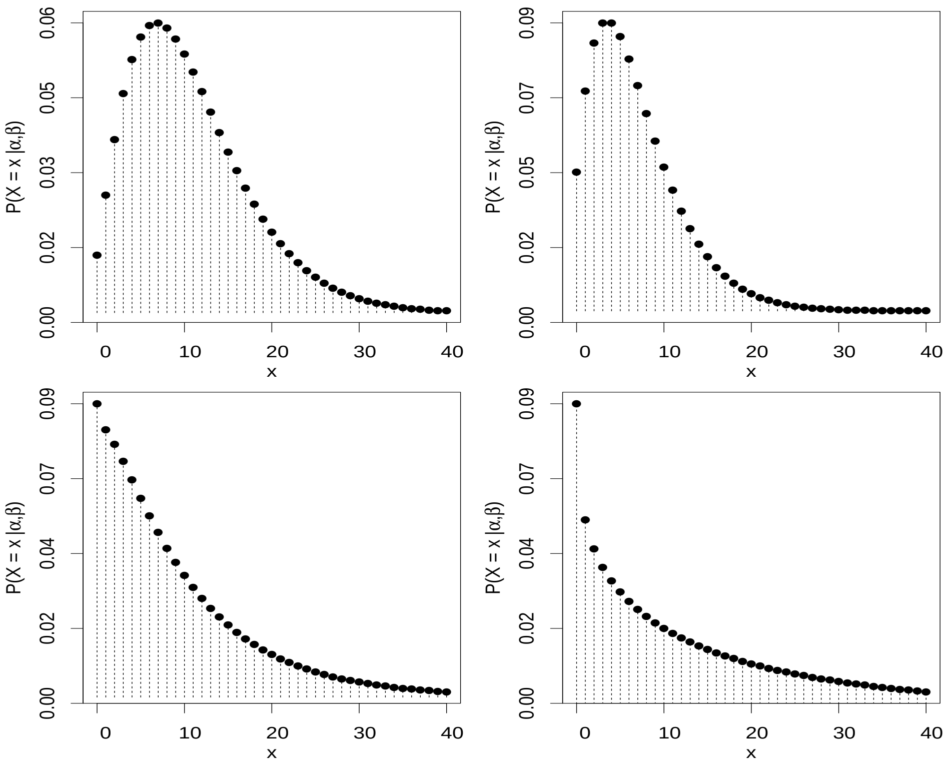

Figure 1 - Behavior of the probability function of the discrete power Lindley distribution considering different values for α and β (upper-left panel: $\alpha = 1.2, \beta = 0.1$, upper-right panel: $\alpha = 1.1, \beta = 0.2$, lower-left panel: $\alpha = 0.8, \beta = 0.3$ and lower-right panel: $\alpha = 0.6, \beta = 0.3$).

The cumulative distribution function and survival function of DPL distribution are given, respectively, as,

$$F(x \mid \alpha, \beta) = 1 - \left(1 + \frac{\beta x^{\alpha}}{\beta + 1}\right) \gamma^{x^{\alpha}}, \tag{7}$$

and,

$$S(x \mid \alpha, \beta) = \left(1 + \frac{\beta x^{\alpha}}{\beta + 1}\right) \gamma^{x^{\alpha}} \tag{8}$$

where $x = 0, 1, 2, \ldots, \alpha, \beta > 0$ and $\gamma = \exp(-\beta)$. Notice that the survival function is the same for the continuous power Lindley and DPL in integer points of x.

2.1 Hazard function

From the equations (6) and (8), it is obtained the hazard rate function which the behavior is illustrated in Figure 2 as follows,

$$h(x \mid \alpha, \beta) = 1 - \left[\frac{1+\beta+\beta(x+1)^{\alpha}}{1+\beta+\beta x^{\alpha}}\right] \gamma^{(x+1)^{\alpha}-x^{\alpha}} \tag{9}$$

where $x = 0, 1, 2, \ldots$, $\alpha, \beta > 0$ and $\gamma = \exp(-\beta)$. Note that, for $x \to 0$ and $x \to \infty$, the hazard hate function became into,

$$h(0 \mid \alpha, \beta) = 1 - \frac{1+2\beta}{1+\beta}\gamma \qquad h(\infty \mid \alpha, \beta) = \begin{cases} 0, \text{ if } \alpha < 1 \\ 1-\gamma, \text{ if } \alpha = 1 \\ 1, \text{ if } \alpha > 1. \end{cases} \tag{10}$$

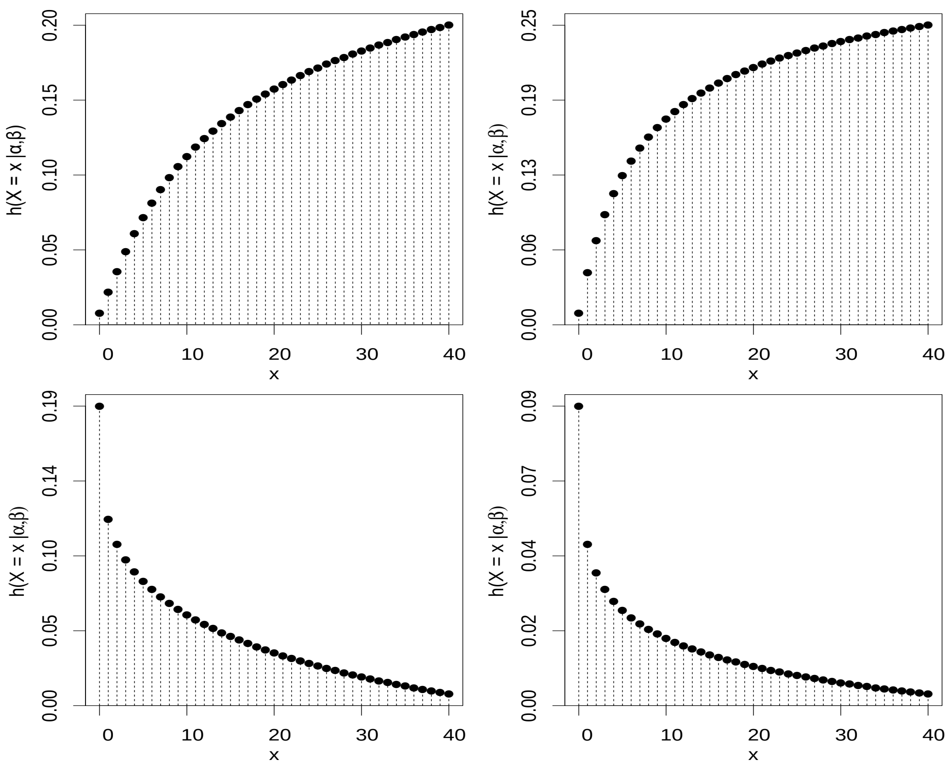

Figure 2 - Behavior of the hazard function of the discrete power Lindley distribution considering different values for α and β (upper-left panel: $\alpha = 0.7, \beta = 0.5$, upper-right panel: $\alpha = 1.2, \beta = 0.1$, lower-left panel: $\alpha = 1.1, \beta = 0.2$ and lower-right panel: $\alpha = 0.6, \beta = 0.3$).

2.2 Quantile

For all $\alpha, \beta > 0$, the quantile function, as in the continuous case, can be written in terms of the Lambert W function:

$$Q(p \mid \alpha, \beta) = \left\lfloor \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}(-(1+\beta)e^{(-1-\beta)}(1-p)) \right]^{\frac{1}{\alpha}} \right\rfloor, \quad 0 < p < 1 \quad (11)$$

where W_{-1} is the lower branch of the Lambert W function and $\lfloor \cdot \rfloor$ denotes the floor of $Q(p \mid \alpha, \beta)$. From (11) and taking $p = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$, the 25%, 50% and 75% percentiles are written as,

$$Q\left(\frac{1}{4} \mid \alpha, \beta\right) = \left\lfloor \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(-\frac{3}{4}(1+\beta)e^{(-1-\beta)} \right) \right]^{\frac{1}{\alpha}} \right\rfloor,$$

$$Q\left(\frac{1}{2} \mid \alpha, \beta\right) = \left\lfloor \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(-\frac{1}{2}(1+\beta)e^{(-1-\beta)} \right) \right]^{\frac{1}{\alpha}} \right\rfloor,$$

$$Q\left(\frac{3}{4} \mid \alpha, \beta\right) = \left\lfloor \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(-\frac{1}{4}(1+\beta)e^{(-1-\beta)} \right) \right]^{\frac{1}{\alpha}} \right\rfloor.$$

2.3 Moments

Let X be a discrete random variable following a DPL distribution. The *kth*-order moment of X could be expressed in an infinite sum as:

$$\begin{aligned} \mathbb{E}[X^k] &= \sum_{x=0}^{\infty} x^k \gamma^{x^\alpha} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x^k \frac{[-\beta(x+1)^\alpha]^j}{j!} \\ &+ \frac{\beta}{\beta+1}\left[\sum_{x=0}^{\infty} x^{k+\alpha}\gamma^{x^\alpha} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x^k (x+1)^\alpha \frac{[-\beta(x+1)^\alpha]^j}{j!} \right]. \quad (12) \end{aligned}$$

Notice that the *kth*-order moment of X does not have a closed form. However, it could be approximated using numerical methods. If $\frac{k}{\alpha} \in \mathbb{N}$ then $\mathbb{E}[X^k]$ reduces to:

$$\begin{aligned} \mathbb{E}[X^k] &= K\left(\frac{k}{\alpha}, \frac{1}{\gamma}\right) - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x^k \frac{[-\beta(x+1)^\alpha]^j}{j!} \\ &+ \frac{\beta}{\beta+1}\left[K\left(\frac{k}{\alpha}+1, \frac{1}{\gamma}\right) - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x^k (x+1)^\alpha \frac{[-\beta(x+1)^\alpha]^j}{j!} \right] \quad (13) \end{aligned}$$

where, from Zwillinger (2014), K is written as,

$$K(a,b) = \frac{1}{(b-1)^{a+1}} \sum_{i=1}^{a} \left[\frac{1}{b^{a-i}} \sum_{j=0}^{i} \frac{(-1)^j (a+1)!(i-j)^a}{j!(a+1-j)!} \right] \qquad a = 1, 2, \ldots, b \neq 1.$$

and the second and fourth terms are exponential series which is convergent for $\alpha, \beta > 0$. From (12) the expressions $\mathbb{E}[X]$ and $\mathrm{Var}[X]$ are, respectively,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{x=0}^{\infty} x\gamma^{x^{\alpha}} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x\frac{[-\beta(x+1)^{\alpha}]^{j}}{j!} \\ &+ \frac{\beta}{\beta+1}\left[\sum_{x=0}^{\infty} x^{1+\alpha}\gamma^{x^{\alpha}} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x(x+1)^{\alpha}\frac{[-\beta(x+1)^{\alpha}]^{j}}{j!}\right] \qquad (14)\end{aligned}$$

and,

$$\begin{aligned}\mathrm{Var}[X] &= \sum_{x=0}^{\infty} x^{2}\gamma^{x^{\alpha}} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x^{2}\frac{[-\beta(x+1)^{\alpha}]^{j}}{j!} \\ &+ \frac{\beta}{\beta+1}\left[\sum_{x=0}^{\infty} x^{2+\alpha}\gamma^{x^{\alpha}} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x^{2}(x+1)^{\alpha}\frac{[-\beta(x+1)^{\alpha}]^{j}}{j!}\right] \\ &- \left\{\sum_{x=0}^{\infty} x\gamma^{x^{\alpha}} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x\frac{[-\beta(x+1)^{\alpha}]^{j}}{j!}\right. \\ &+ \left.\frac{\beta}{\beta+1}\left[\sum_{x=0}^{\infty} x^{1+\alpha}\gamma^{x^{\alpha}} - \sum_{x=0}^{\infty}\sum_{j=0}^{\infty} x(x+1)^{\alpha}\frac{[-\beta(x+1)^{\alpha}]^{j}}{j!}\right]\right\}^{2}. \qquad (15)\end{aligned}$$

The dispersion index of DPL distribution, $DI[X] = \mathrm{Var}[X]/\mathbb{E}[X]$, is presented in Table 1. From the results, it is concluded that the DPL distribution could be used for underdispersion when $\alpha, \beta > 1$ or overdispersion when $\alpha, \beta \leq 1$ data modeling. In Figure 3, it is also illustrated the behavior of the expected value and the variance of DPL distribution for some values of α and $\beta = 3$.

Table 1 - Expected value, variance and DI of DPL distribution

(α, β) values	$\mathbb{E}[X]$	$\mathrm{Var}[X]$	$\mathrm{DI}[X]$
(0.5, 0.5)	18.202	1056.241	58.027
(1.0, 0.5)	2.847	7.544	2.649
(0.5, 1.0)	3.602	55.284	15.348
(1.0, 1.0)	1.042	1.705	1.636
(1.5, 1.5)	0.397	0.324	0.815
(2.0, 1.5)	0.365	0.249	0.681

2.4 Order statistics

Considering $X_1, \ldots, X_n$ a random sample of the DPL distribution with parameters α and β and $X_{1:n}, \ldots, X_{n:n}$ the sample order statistics, the probability

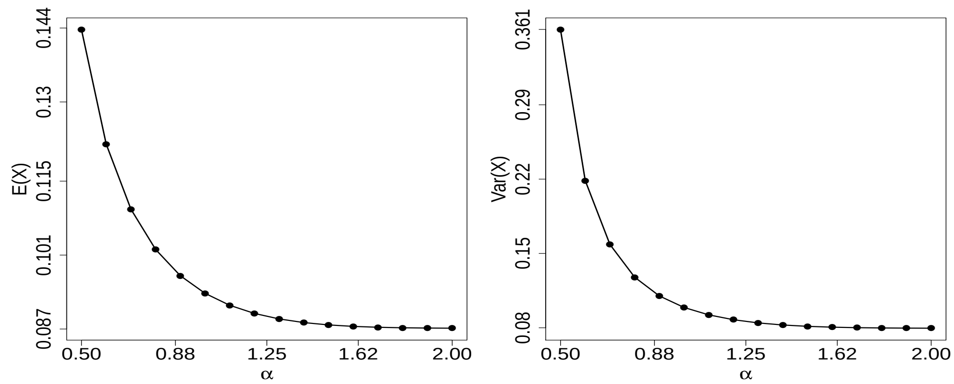

Figure 3 - Behavior of the expected value and the variance the discrete power Lindley distribution considering different values for α and $\beta = 3$ (left panel: Expected value; right panel: Variance).

function and the cumulative function of the *ith*-order statistics $X_{i:n}$ are described, respectively, by the equations:

$$\begin{aligned} P(X_{i:n} = x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)(1+\beta)^{i+j}} \\ &\times \{[(1+\beta)-(1+\beta+\beta x^{\alpha})\gamma^{x^{\alpha}}]^{i+j} \\ &- [(1+\beta)-(1+\beta+\beta(x-1)^{\alpha})\gamma^{(x-1)^{\alpha}}]^{i+j}\} \end{aligned}$$

and,

$$P(X_{i:n} < x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j[(1+\beta)-(1+\beta+\beta x^{\alpha})\gamma^{x^{\alpha}}]^{i+j}}{(i+j)(1+\beta)^{i+j}}.$$

Moreover, the correspondent *kth*-order moment of the *ith*-order statistics is expressed as:

$$\begin{aligned} \mathbb{E}[X_{i:n}^k] &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)(1+\beta)^{i+j}} \\ &\times\{\sum_{m=0}^{i+j}\binom{i+j}{m}(1+\beta)^{i+j-m}(-1)^m\sum_{l=0}^{m}\binom{m}{l}(1+\beta)^{m-l}\beta^l\sum_{x=0}^{\infty}x^{(k/\alpha)+l}\gamma^{mx^{\alpha}} \\ &-\sum_{m=0}^{i+j}\binom{i+j}{m}(1+\beta)^{i+j-m}(-1)^m\sum_{l=0}^{m}\binom{m}{l}(1+\beta)^{m-l}\beta^l\sum_{x=0}^{\infty}x^k(x-1)^l\gamma^{m(x-1)^{\alpha}}\}. \end{aligned}$$

3 Maximum likelihood estimation

In this section, we will derive the maximum likelihood function for the model presented in the previous section. Through the frequentist approach, the likelihood function can used to obtain point estimates for the parameters α and β of the proposed model. Moreover, suitable estimates for the confidence intervals can be obtained using large sample approximations, that are based on the asymptotic properties of the maximum likelihood estimators.

3.1 Inference under DPL distribution

Considering $x_1, \ldots, x_n$ a random sample of the DPL distribution, with parameters α and β, and probability function given by (6) we have the likelihood function written as:

$$L(\alpha, \beta \mid \mathbf{x}) = \prod_{i=1}^{n} \left[\left(1 + \frac{\beta x_i^{\alpha}}{\beta+1}\right) \gamma^{x_i^{\alpha}} - \left(1 + \frac{\beta (x_i+1)^{\alpha}}{\beta+1}\right) \gamma^{(x_i+1)^{\alpha}} \right]. \quad (16)$$

From (16), the log-likelihood can be written as:

$$\ell(\alpha, \beta \mid \mathbf{x}) = -n \ln(\beta+1) + \sum_{i=1}^{n} \ln \left[(1+\beta+\beta x_i^{\alpha})\gamma^{x_i^{\alpha}} - (1+\beta+\beta(x_i+1)^{\alpha})\gamma^{(x_i+1)^{\alpha}} \right] \quad (17)$$

which is maximized solving numerically, in α and β, the non-linear system of the equations,

$$U_n = \begin{cases} \dfrac{\partial \ell}{\partial \alpha} = \displaystyle\sum_{i=1}^{n} \dfrac{\beta \left[(1+\beta+\beta\,(x_i+1))\, \gamma^{(x_i+1)^{\alpha}}\, (x_i+1)^{\alpha} \ln(x_i+1) \right]}{(\beta\, x_i + \beta + 1)\, \gamma^{x_i{}^{\alpha}} - (1+\beta+\beta\,(x_i+1))\, \gamma^{(x_i+1)^{\alpha}}} \\ \qquad - \dfrac{\beta \left[(\beta\, x_i + \beta + 1)\, \gamma^{x_i{}^{\alpha}} x_i{}^{\alpha} \ln(x_i) \right]}{(\beta\, x_i + \beta + 1)\, \gamma^{x_i{}^{\alpha}} - (1+\beta+\beta\,(x_i+1))\, \gamma^{(x_i+1)^{\alpha}}} \\ \dfrac{\partial \ell}{\partial \beta} = -\dfrac{n}{\beta+1} + \displaystyle\sum_{i=1}^{n} \dfrac{(x_i+1)\, \gamma^{x^{\alpha}} - (x_i+2)\, \gamma^{(x_i+1)^{\alpha}}}{(\beta\, x_i + \beta + 1)\, \gamma^{x_i{}^{\alpha}} - (1+\beta+\beta\,(x_i+1))\, \gamma^{(x_i+1)^{\alpha}}} \end{cases} \quad (18)$$

There is no closed form for the MLE of α and β, see (18). However, using (17) one can estimate α and β using standard numeric optimization algorithms such the Newton-Raphson or Nelder-Mead methods. By the usual maximum likelihood theory, an asymptotic approximation for the variance of $\widehat{\alpha}$ and $\widehat{\beta}$ can be obtained from U_n^{-1}, which evaluated at $\widehat{\alpha}$ and $\widehat{\beta}$ provides a consistent estimator for such a measure.

In order to obtain interval estimates, we can use large sample approximations for the $100 \times (1-\eta)\,\%$ two sided confidence interval (CI), i.e. $\widehat{\alpha} \pm z_{\eta/2}\, \widehat{\mathrm{se}}\,(\widehat{\alpha})$ and $\widehat{\beta} \pm z_{\eta/2}\, \widehat{\mathrm{se}}\left(\widehat{\beta}\right)$, where z_{η} is the upper η^{th} percentile of the standard Normal distribution and the standard error (SE) is estimated as the squared root of the variance of $\widehat{\alpha}$ and $\widehat{\beta}$.

4 Simulation study

In this section we estimated, by Monte Carlo simulation, the biases and the mean squared errors for the maximum likelihood estimators for $\widehat{\alpha}$ and $\widehat{\beta}$. We adopted $\alpha \times \beta = (0.5, 1.0, 1.5) \times (0.5, 1.0, 1.5)$ and sample sizes $n = 20, 40, \ldots, 200$. For each scenario, we had calculated:

$$\text{BIAS}(\widehat{\theta}) = \frac{1}{B}\sum_{i=1}^{B}(\widehat{\theta}_i - \theta_i) \quad \text{and} \quad \text{MSE}(\widehat{\theta}) = \frac{1}{B}\sum_{i=1}^{B}(\widehat{\theta}_i - \theta_i)^2$$

where $\theta = (\alpha, \beta)$ and $B = 10.000$ replications. The inverse-transform method for discrete distributions was implemented to generate the pseudo-random samples. The simulation process was performed using R software.

Tables 2 and 3 show the simulation results. In every scenario, the bias of $\widehat{\alpha}$ is positive and tends to zero when the sample size increases. The bias of $\widehat{\beta}$, even when oscillating between positive and negative, tends to zero when the sample size increases in every scenario. The mean squared error of $\widehat{\alpha}$ and $\widehat{\beta}$ tends to zero in every scenario. In Figure 4 it is also illustrated the behavior of estimated the biases and estimated mean-squared-errors of maximum likelihood estimators of the DPL distribution.

Table 2 - Estimated bias and mean squared error for $\widehat{\alpha}$

	n	$\beta = 0.5$			$\beta = 1.0$			$\beta = 1.5$		
		α			α			α		
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
BIAS	20	0.0331	0.0668	0.1052	0.0442	0.0942	0.2262	0.0921	0.3254	0.9583
	40	0.0171	0.0294	0.0518	0.0208	0.0407	0.0755	0.0330	0.0911	0.4669
	60	0.0124	0.0188	0.0329	0.0137	0.0271	0.0459	0.0208	0.0501	0.2303
	80	0.0101	0.0139	0.0249	0.0098	0.0202	0.0348	0.0153	0.0367	0.1286
	100	0.0089	0.0108	0.0198	0.0077	0.0166	0.0286	0.0124	0.0287	0.0806
	120	0.0080	0.0094	0.0161	0.0069	0.0136	0.0235	0.0101	0.0241	0.0550
	140	0.0077	0.0077	0.0137	0.0058	0.0121	0.0192	0.0085	0.0212	0.0415
	160	0.0071	0.0069	0.0121	0.0050	0.0111	0.0164	0.0073	0.0186	0.0337
	180	0.0067	0.0063	0.0109	0.0045	0.0099	0.0141	0.0063	0.0166	0.0292
	200	0.0065	0.0054	0.0101	0.0037	0.0093	0.0127	0.0059	0.0149	0.0264
MSE	20	0.0099	0.0420	0.1036	0.0179	0.0838	0.4539	0.0960	0.7393	2.3946
	40	0.0041	0.0168	0.0423	0.0069	0.0283	0.0869	0.0137	0.1179	1.1408
	60	0.0026	0.0105	0.0259	0.0043	0.0178	0.0460	0.0079	0.0371	0.5220
	80	0.0019	0.0077	0.0189	0.0030	0.0129	0.0321	0.0055	0.0250	0.2602
	100	0.0015	0.0061	0.0148	0.0024	0.0102	0.0253	0.0042	0.0189	0.1394
	120	0.0012	0.0050	0.0123	0.0019	0.0083	0.0202	0.0034	0.0154	0.0798
	140	0.0010	0.0043	0.0104	0.0017	0.0071	0.0168	0.0029	0.0130	0.0538
	160	0.0009	0.0037	0.0090	0.0014	0.0061	0.0145	0.0025	0.0110	0.0401
	180	0.0008	0.0033	0.0080	0.0013	0.0055	0.0128	0.0022	0.0096	0.0330
	200	0.0007	0.0029	0.0072	0.0011	0.0049	0.0114	0.0020	0.0085	0.0263

Table 3 - Estimated bias and mean squared error for $\widehat{\beta}$

	n	$\beta = 0.5$			$\beta = 1.0$			$\beta = 1.5$		
		α			α			α		
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
BIAS	20	-0.0079	-0.0092	-0.0104	0.0036	0.0019	0.0080	0.0293	0.0414	0.0468
	40	-0.0046	-0.0034	-0.0058	0.0003	0.0014	0.0023	0.0093	0.0155	0.0211
	60	-0.0040	-0.0023	-0.0036	-0.0003	0.0010	0.0023	0.0042	0.0084	0.0136
	80	-0.0034	-0.0016	-0.0027	-0.0002	0.0000	0.0012	0.0037	0.0068	0.0106
	100	-0.0034	-0.0013	-0.0019	-0.0002	-0.0008	0.0003	0.0033	0.0048	0.0082
	120	-0.0032	-0.0015	-0.0015	-0.0007	-0.0005	0.0008	0.0038	0.0037	0.0066
	140	-0.0035	-0.0010	-0.0013	-0.0004	-0.0007	0.0013	0.0036	0.0035	0.0064
	160	-0.0031	-0.0009	-0.0014	-0.0004	-0.0008	0.0011	0.0032	0.0024	0.0059
	180	-0.0031	-0.0010	-0.0012	-0.0004	-0.0007	0.0014	0.0029	0.0019	0.0053
	200	-0.0032	-0.0006	-0.0012	0.0003	-0.0007	0.0012	0.0022	0.0013	0.0046
MSE	20	0.0204	0.0209	0.0222	0.0594	0.0613	0.0638	0.5149	0.1676	0.1365
	40	0.0100	0.0101	0.0107	0.0285	0.0286	0.0292	0.0589	0.0597	0.0600
	60	0.0067	0.0066	0.0070	0.0186	0.0186	0.0192	0.0376	0.0385	0.0391
	80	0.0050	0.0050	0.0053	0.0137	0.0140	0.0145	0.0278	0.0284	0.0292
	100	0.0040	0.0041	0.0042	0.0109	0.0112	0.0114	0.0225	0.0222	0.0230
	120	0.0033	0.0034	0.0036	0.0090	0.0093	0.0094	0.0186	0.0185	0.0191
	140	0.0029	0.0029	0.0031	0.0077	0.0079	0.0080	0.0159	0.0161	0.0164
	160	0.0025	0.0026	0.0027	0.0068	0.0069	0.0070	0.0140	0.0141	0.0144
	180	0.0022	0.0023	0.0024	0.0060	0.0061	0.0062	0.0123	0.0126	0.0127
	200	0.0020	0.0020	0.0021	0.0054	0.0056	0.0056	0.0110	0.0114	0.0114

5 Applications

In this section, the DPL distribution is considered as an attempt to adequately model two datasets from different areas. The goodness-of-fit of the proposed model is compared with those accessed by the Poisson (P), the discrete Lindley (DL) and the discrete Burr XII (BURR) distributions in the application one; and the discrete half-normal (DHN), Poisson (P) and the discrete Lindley (DL) in the application two. Both applications was done in R software (R CORE TEAM, 2016).

As first application, let us consider the related data to the number of industrial accidents of 647 women that worked in reservoirs with explosion hazard, in a period of five weeks (see Greenwood and Wood, 1919). The mean and variance are, respectively, $\overline{x} = 0.46$ accidents and $s^2 = 0.69$ accidents^2, which evidences over-dispersion. The fit of DPL distribution was compared to the fit of discrete Burr XII distribution, $P(X = x \mid \beta, \alpha) = \beta^{\log(1+x^{\alpha})} - \beta^{\log(1+(1+x)^{\alpha})}$, discrete Lindley, $P(X = x \mid \beta) = e^{-\beta x}(1+\beta)^{-1}[\beta(1-e^{-\beta}) + (1-e^{-\beta})(1+\beta x)]$, and Poisson, $P(X = x \mid \beta) = \frac{e^{-\beta}\beta^{x}}{x!}$.

In Table 4, we present the frequency distribution of each sample. The expected frequencies was obtained using the estimated probabilities considering the MLEs of the parameters. Frequencies in bold relate to those one closer to the observed ones. The results show that the DPL distribution provide good fit to the number of industrial accidents data set.

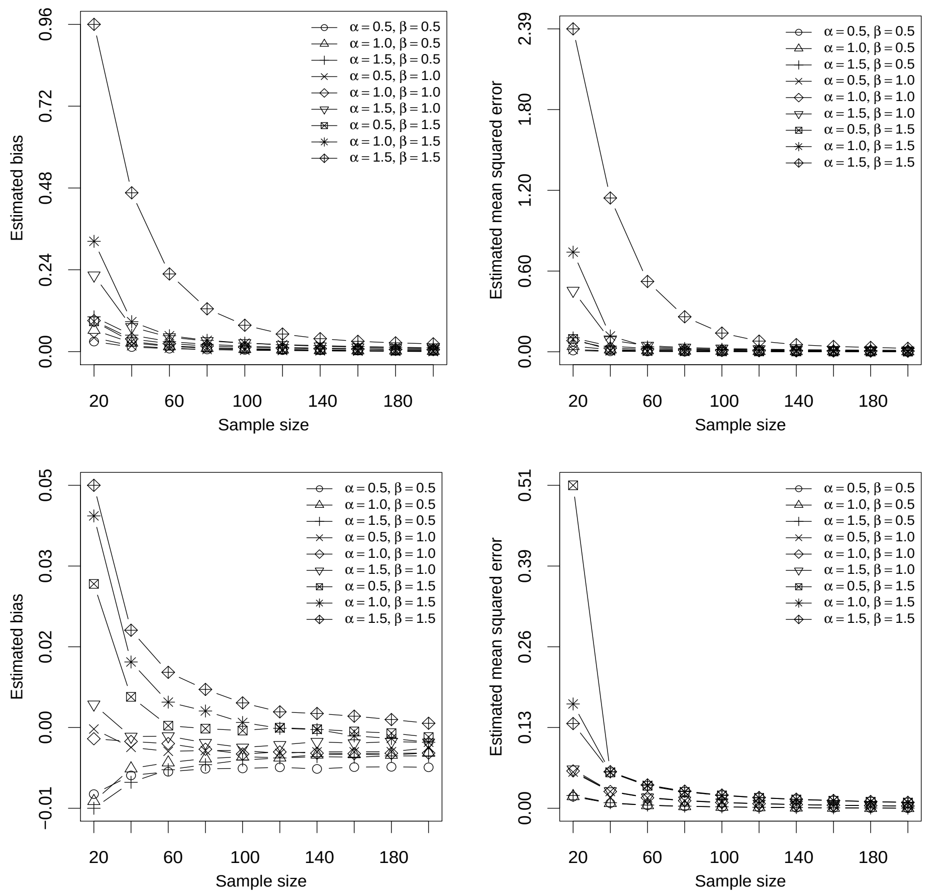

Figure 4 - Behavior of estimated the biases and estimated mean-squared-errors of maximum likelihood estimators of the DPL distribution (upper-left panel: Estimated BIAS of $\widehat{\alpha} \times n$, upper-right panel: Estimated MSE of $\widehat{\alpha} \times n$, lower-left panel: Estimated BIAS of $\widehat{\beta} \times n$ and lower-right panel: Estimated MSE of $\widehat{\beta} \times n$).

The MLEs, the SEs, and the goodness-of-fit measures are presented in Table 5. The model selection was performed using the Akaike information criterion and the Bayesian information criterion (BIC). Notice that the smaller AIC's are provided by the DPL distribution. The goodness-of-fit was evaluated by the χ^2 statistic. In this case, the chi-square value with 3 d.f. for the DPL distribution is $\chi^2 = 3.77$, with corresponding p-value equals to 0.286, highlighting the adherence of the DPL distribution. For the other models, the adherence hypothesis was rejected.

Table 4 - Observed and expected number of accidents considering the DPL, BURR, DL and P distributions

N^o of accidents	Observed Frequency	Expected Frequency			
		DPL	BURR	DL	P
0	447	445.63	**446.72**	430.05	406.31
1	132	**135.03**	141.37	154.65	189.02
2	42	44.25	35.30	45.75	**43.96**
3	21	**14.65**	11.96	12.34	6.81
4+	5	7.42	11.63	**4.17**	0.86

Table 5 - Parameter estimates and goodness-of-fit measures

Model	MLE	S.E.	$-\log L$	χ^2	p-value	D.F.	AIC	BIC
P	$\widehat{\beta} = 0.47$	0.03	617.18	103.13	0.001	4	1236.37	1240.84
DL	$\widehat{\theta} = 1.57$	0.05	595.28	11.29	0.023	4	1192.57	**1197.04**
BURR	$\widehat{\theta} = 0.18$	0.01	597.95	12.74	0.005	3	1199.91	1208.86
	$\widehat{\alpha} = 1.64$	0.12						
DPL	$\widehat{\alpha} = 0.88$	0.05	**592.26**	3.77	**0.286**	3	**1188.53**	1197.47
	$\widehat{\beta} = 1.65$	0.06						

As second application, it is considered the data related to the number of red European mites over the apple tree's leaves (see, Bliss and Fisher (1953); Consul and Jain (1973)). The mean and variance are, respectively, $\overline{x} = 1.14$ mites and $s^2 = 2.25$ mites^2. As alternatives of DPL distribution we considered the discrete Lindley distribution, the Poisson distribution and the discrete half-normal distribution with pmf given by $P(X = x \mid \sigma) = 2[\Phi_\sigma(x+1) - \Phi_\sigma(x)]$(GÓMEZ-DÉNIZ *et. al.*, 2014).

In Table 6, we present the frequency distribution of each sample. The expected frequencies was obtained using the estimated probabilities considering the MLEs of the parameters. Frequencies in bold relate to those one closer to the observed ones. The results show that the DPL distribution provide a good fit to the number of red European mites over the apple tree's leaves data set.

The MLEs, the SEs, and the goodness-of-fit measures are presented in Table 7. The model selection was performed using the Akaike information criterion and the Bayesian information criterion (BIC). Notice that the smaller AIC's are provided by the DPL distribution. The goodness-of-fit was evaluated by the χ^2 statistic. In this case, the chi-square value with 3 d.f. for the DPL distribution is $\chi^2 = 2.53$, with corresponding p-value equals to 0.469, highlighting the adherence of the DPL distribution. Except for DL and DPL models, the adherence hypothesis was rejected for the other models.

Table 6 - Observed and expected number of red European mites considering the DPL, DL, DHN and P distributions

N^o of mites	Observed Frequency	Expected Frequency			
		DPL	DL	DHN	P
0	70	**69.34**	63.13	52.35	47.65
1	38	**37.48**	41.93	42.81	54.64
2	17	**20.32**	23.10	28.63	31.33
3	10	**10.81**	11.64	15.66	11.97
4	9	5.71	5.56	**7.00**	3.43
5+	6	**6.33**	4.64	3.56	0.97

Table 7 - Parameter estimates and goodness-of-fit measures

Model	MLE	S.E.	$-\log L$	χ^2	p-value	D.F.	AIC	BIC
DHN	$\widehat{\sigma} = 2.12$	0.13	229.12	15.52	0.003	4	460.26	463.27
P	$\widehat{\lambda} = 1.15$	0.09	242.81	57.67	< 0.001	4	487.62	490.63
DL	$\widehat{\theta} = 0.94$	0.05	223.63	5.47	0.241	4	449.26	**452.27**
DPL	$\widehat{\alpha} = 0.88$	0.07	**222.44**	2.53	**0.469**	3	**448.89**	454.91
	$\widehat{\beta} = 1.03$	0.08						

6 Conclusion

In this paper, the discrete power Lindley distribution was introduced as alternative to model count data. To derive the proposed model, it is considered the discretization method based on the survival function. Some mathematical properties as the pmf behavior, hazard function, $k-th$ moment and order statistics were discussed. Moreover, it was performed a Monte Carlo simulation study where the bias and the mean squared errors were computed and indicated that the parameters are asymptotically non-biased and consistent. The usefulness of the proposed model was evaluated by fitting it to two count datasets. The model selection was performed using the AIC and the BIC criteria. The goodness-of-fit was accessed by the χ^2 statistic. The provided results illustrated that the DPL distribution has a good fit and could compete with standard models or new proposed discrete models.

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- *RESUMO: Métodos para gerar uma distribuição discreta análoga a uma distribuição contíinua têm sido amplamente considerados nas últimas décadas. Em geral, o procedimento de discretização compreende transformar atributos contínuos em atributos discretos, gerando novas distribuições de probabilidade que poderiam ser uma alternativa aos tradicionais modelos discretos, como os modelos Poisson e Binomial, comumente usados na análise de dados de contagem. Tambéem evita o uso contínuo na análise de dados estritamente discretos. Neste trabalho, utilizando o método de discretização baseado na função de sobrevivência, é introduzido um análogo discreto da distribuição de potência de Lindley. Algumas propriedades matemáticas são estudadas. A teoria da máxima verossimilhança é considerada para estimativas e inferência assintótica. Um estudo de simulação também é realizado para avaliar algumas propriedades dos estimadores de máxima verossimilhança do modelo proposto. A utilidade e exatidão do modelo proposto é avaliada usando conjuntos de dados reais fornecidos pela literatura.*
- *PALAVRAS-CHAVE: Discretizaçao; distribuição potência de Lindley; simulação de Monte Carlo; estimadores de máxima verossimilhança.*

References

BAKOUCH, H. S.; JAZI, M. A.; NADARAJAH, S. A new discrete distribution. *Statistics*, Taylor & Francis, v. 48, n. 1, p. 200–240, 2014.

BARCO, K. V. P.; MAZUCHELI, J.; JANEIRO, V. The inverse power Lindley distribution. *Communications in Statistics-Simulation and Computation*, Taylor & Francis, p. 1–16, 2017.

BLISS, C. I.; FISHER, R. A. Fitting the negative binomial distribution to biological data. *Biometrics*, JSTOR, v. 9, n. 2, p. 176–200, 1953.

BOULLE, M. Khiops: A statistical discretization method of continuous attributes. *Machine learning*, Springer, v. 55, n. 1, p. 53–69, 2004.

BRACQUEMOND, C.; GAUDOIN, O. A survey on discrete lifetime distributions. *International Journal of Reliability, Quality and Safety Engineering*, World Scientific, v. 10, n. 01, p. 69–98, 2003.

CHAKRABORTY, S. Generating discrete analogues of continuous probability distributions-a survey of methods and constructions. *Journal of Statistical Distributions and Applications*, Springer, v. 2, n. 1, p. 1–30, 2015.

CHAKRABORTY, S.; CHAKRAVARTY, D. Discrete gamma distributions: properties and parameter estimations. *Communications in Statistics-Theory and Methods*, Taylor & Francis, v. 41, n. 18, p. 3301–3324, 2012.

CONSUL, P. C.; JAIN, G. C. A generalization of the Poisson distribution. *Technometrics*, Taylor & Francis Group, v. 15, n. 4, p. 791–799, 1973.

GHITANY, M.; AL-MUTAIRI, D.; NADARAJAH, S. Zero-truncated Poisson–Lindley distribution and its application. *Mathematics and Computers in Simulation*, Elsevier, v. 79, n. 3, p. 279–287, 2008.

GHITANY, M. et al. A two-parameter weighted Lindley distribution and its applications to survival data. *Mathematics and Computers in Simulation*, Elsevier, v. 81, n. 6, p. 1190–1201, 2011.

GHITANY, M. E. et al. Power Lindley distribution and associated inference. *Computational Statistics & Data Analysis*, Elsevier, v. 64, p. 20–33, 2013.

GHOSH, T.; ROY, D.; CHANDRA, N. K. Reliability approximation through the discretization of random variables using reversed hazard rate function. *International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering*, International Science Index, v. 7, n. 4, p. 96 – 100, 2013.

GÓMEZ-DÉNIZ, E.; CALDERÍN-OJEDA, E. The discrete Lindley distribution: properties and applications. *Journal of Statistical Computation and Simulation*, Taylor & Francis, v. 81, n. 11, p. 1405–1416, 2011.

GÓMEZ-DÉNIZ, E.; VÁZQUEZ-POLO, F.; GARCÍA-GARCÍA, V. A discrete version of the half-normal distribution and its generalization with applications. *Statistical Papers*, Springer, v. 55, n. 2, p. 497–511, 2014.

GOOD, I. J. The population frequencies of species and the estimation of population parameters. *Biometrika*, Biometrika Trust, v. 40, n. 3-4, p. 237–264, 1953.

GREENWOOD, M.; WOODS, H. M. *The Incidence of Industrial Accidents Upon Individuals: With Special Reference to Multiple Accidents*. [S.l.]: HM Stationery Office [Darling and son, Limited, printers], 1919.

HUSSAIN, T.; AHMAD, M. Discrete inverse Rayleigh distribution. *Pak. J. Statist*, v. 30, n. 2, p. 203–222, 2014.

JAZI, M. A.; LAI, C.-D.; ALAMATSAZ, M. H. A discrete inverse Weibull distribution and estimation of its parameters. *Statistical Methodology*, Elsevier Science, v. 7, p. 121–132, 2010.

KEILSON, J.; GERBER, H. Some results for discrete unimodality. *Journal of the American Statistical Association*, American Statistical Association, v. 66, 1971.

KEMP, A. W. Characterizations of a discrete normal distribution. *Journal of Statistical Planning and Inference*, v. 63, n. 2, p. 223 – 229, 1997.

KEMP, A. W. Classes of discrete lifetime distributions. Taylor & Francis, 2004.

KRISHNA, H.; PUNDIR, P. S. Discrete Burr and discrete Pareto distributions. *Statistical Methodology*, Elsevier, v. 6, n. 2, p. 177–188, 2009.

KULASEKERA, K.; TONKYN, D. W. A new discrete distrlbution, with applications to survival, dispersal and dispersion. *Communications in Statistics-Simulation and Computation*, Taylor & Francis, v. 21, n. 2, p. 499–518, 1992.

LINDLEY, D. V. Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, v. 20, p. 102–107, 1958.

MEROVCI, F. Transmuted Lindley distribution. *International Journal of Open Problems in Computer Science & Mathematics*, v. 6, 2013.

NAKAGAWA, T.; OSAKI, S. The discrete Weibull distribution. *IEEE Transactions on Reliability*, v. 5, p. 300–301, 1975.

NEKOUKHOU, V.; ALAMATSAZ, M. H.; BIDRAM, H. Discrete generalized exponential distribution of a second type. *Statistics*, Taylor and Francis Group, v. 47, p. 876–887, 2013.

R Core Team. *R: A Language and Environment for Statistical Computing*. Vienna, Austria, 2016. Disponível em: ⟨https://www.R-project.org/⟩.

ROY, D. The discrete normal distribution. *Communication in Statistics- Theory and Methods*, Taylor and Francis Group, v. 32, p. 1871–1883, 2003.

ROY, D. Discrete Rayleigh distribution. *Reliability, IEEE Transactions on*, IEEE, v. 53, n. 2, p. 255–260, 2004.

ROY, D.; GHOSH, T. A new discretization approach with application in reliability estimation. *IEEE Transactions on Reliability*, IEEE, v. 58, 2009.

SANKARAN, M. The discrete Poisson-Lindley distribution. *Biometrics*, v. 26, p. 145–149, 1970.

SATO, H. et al. A new defect distribution metrology with a consistent discrete exponential formula and its applications. *Semiconductor Manufacturing, IEEE Transactions on*, IEEE, v. 12, n. 4, p. 409–418, 1999.

SHANKER, R.; MISHRA, A. A quasi Lindley distribution. *African Journal of Mathematics and Computer Science Research*, Academic Journals, v. 6, n. 4, p. 64–71, 2013.

SHARMA, V. K. et al. The inverse Lindley distribution: A stress-strength reliability model. 2015.

STEIN, W. E.; DATTERO, R. A new discrete Weibull distribution. *IEEE Transactions on Reliability*, Institute of Electrical and Electronics Engineers, v. 33, n. 2, p. 196–197, 1984.

ZWILLINGER, D. *Table of integrals, series, and products.* [S.l.]: Elsevier, 2014.

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