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Bifurcation analysis of commensalism interaction and harvesting on food chain model¹

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Abstract

In this paper, we study the incorporation of the commensalism interaction and harvesting on the Lotka–Volterra food chain model. The system provides one commensal prey, one harvested prey, and two predators. A set of preliminary results in local bifurcation analysis around each equilibrium point for the proposed model is discussed, such as saddle-node, transcritical and pitchfork. Some numerical analysis to confirm the accruing of local bifurcation is illustrated. To back up the conclusions of the mathematical study, a numerical simulation of the model is carried out with the help of the MATLAB program. It can be concluded that the system's coexistence can be achieved as long as the harvesting rate on the second prey population is lower than its intrinsic growth rate. Further, the role of mutual interaction can lead to the stability of the proposed system.

Keywords: Local Bifurcation, Harvesting, Commensalism interaction, prey-predator model.

1. Introduction

Ecosystems are the result of interactions between the environment and communities. In an ecosystem, a food chain plays a vital role in guaranteeing the stability of the population (Saijnders & Bazin, 1975). The best method to understand the dynamics and behavior of ecological interactions between prey and predator populations is to utilize a mathematical model. A simple model of prey-predator interactions was proposed separately by Lotka and Volterra, but the model is now known as the Lotka – Volterra model (Lotka, 1926; Volterra, 1926). In the literature, Paine (1966) investigated and analyzed the first simple mathematical model of two prey and one predator in terms of predicting their dynamics. Subsequently, researchers studied numerous properties, such as coexistence, persistence, stability and extinction (Abakumov & Izrailsky, 2022; Gard & Hallam, 1979; Shireen Jawad, 2022; Shireen Jawad, Sultan & Winter, 2021; Mortoja, Panja & Mondal, 2018; Paine, 1966; Xue *et al.*, 2015; Dawud & Jawad, 2022). In addition to the above, used Holling type-I functional response in a system consisting of two prey-one predators. (Elettrey, 2009; Colucci, 2013) explored the difficulties in the dynamic behavior of two prey-one predator systems following a Holling type II functional response with an influence impulsive.

Moreover, Ali (2016) analyzed the local and global stability of the prey-predator model, including Holling type I functional response and the implications of group help. Further, Tolcha, Bole & Koya (2020) considered the interaction between two mutualistic prey and a predator population. In addition, the proportional harvesting function is taken into account in his model when these species interact. The stability of his model has been established for the positive equilibrium point.

The mathematical study of changes in a family's qualitative or topological structure, such as the integral curves of a family of vector fields or the solutions to a family of differential equations, is known as bifurcation theory. A bifurcation happens when a system's behaviour abruptly changes in a "qualitative" or topological way due to a tiny, gradual change in its parameter values (the bifurcation parameters), a term most frequently used to describe the mathematical study of dynamical systems (Place, 2017). The term "bifurcation" was initially used by Poincaré (Poincaré, 1885) in 1885 in the first mathematical article to demonstrate this phenomenon. He also went on to describe and categorize several sorts of stationary points. Many scholars examined the bifurcation analysis on their model's behaviour (Collings, 1995; Guckenheimer & Holmes, 2013; Hubbard, Gong & Ruan, 2013; Kar, 2009; Poincaré, 1885; Rong, 1996). For example, Perko (2013) recognized saddle-node, transcritical and pitchfork bifurcation conditions in his work.

On the other hand, species must be harvested in order to provide people with a resource. Numerous researchers have already looked into how harvesting affects the population system (Hu & Cao, 2017; Shireen Jawad, 2022; Kar & Pahari, 2007; Mandal *et al.*, 2021; Al Nuaimi & Jawad, 2022; Perko, 2013). In ecosystems, there are three different harvesting types: constant, linear, and nonlinear. In general, harvesting might cause the system to behave in complicated, dynamic ways. For instance, Idlango, Shepherd & Gear (2017) showed that the logistic model with a Holling type II harvesting element could accept zero, one, or two positive equilibria.

In this paper, we consider the interaction among four populations: two prey and two predators. The first prey is assumed to help the second, whilst the latter is harvested. The first predator can attack the first prey, while the second predator (top predator) can only attack the first predator, according to the type I functional response. The local stability property of the equilibria of the proposed system is investigated. Then local bifurcation conditions are explored. Finally, some numeric simulations are presented to show the feasibility of the main results. We end this paper with a brief discussion.

2. Assumptions of the Model

Suppose a food chain contains the following species: prey, a predator and a top predator, with the mathematics beings based on the following assumptions. $n_1(t)$ is the density of the first prey (the first species in the food chain), $n_2(t)$ is the density of the second harvested prey, which has a positive effect on the first prey, whilst $n_3(t)$ and $n_4(t)$ are the densities of the predator and top predator species, respectively. Under the above assumptions, the model can be presented by the following system of differential equations:

$$\begin{aligned} \frac{dn_1}{dt} &= rn_1 \left(1 - \frac{n_1}{k}\right) - \beta_1 n_1 n_3 + an_1 n_2 = n_1 f_1(n_1, n_2, n_3, n_4) \\ \frac{dn_2}{dt} &= sn_2 \left(1 - \frac{n_2}{l}\right) - qEn_2 = n_2 f_2(n_1, n_2, n_3, n_4) \\ \frac{dn_3}{dt} &= \beta_2 n_1 n_3 - \beta_0 n_3 - \gamma_1 n_3 n_4 = n_3 f_3(n_1, n_2, n_3, n_4) \\ \frac{dn_4}{dt} &= \gamma_2 n_3 n_4 - \alpha n_4 = n_4 f_4(n_1, n_2, n_3, n_4). \end{aligned} \tag{1}$$

All parameters of the system (1) are assumed to be positive and described as: k and l , are the carrying capacities of the first and second prey, respectively, with intrinsic growth rates r and s ; a is the positive effect on the first prey by the second prey; E, q are the effort and the catchability rate applied on the second prey, i.e., qE represents the harvesting rate of the second prey; β_1 and γ_1 are the attack rate coefficient of the first prey and first predator species due to the first predator and top predator, respectively; β_0 and α represent the first and the second predator's natural death rates, respectively. The flow chart of the proposed system is shown in the following block diagram.

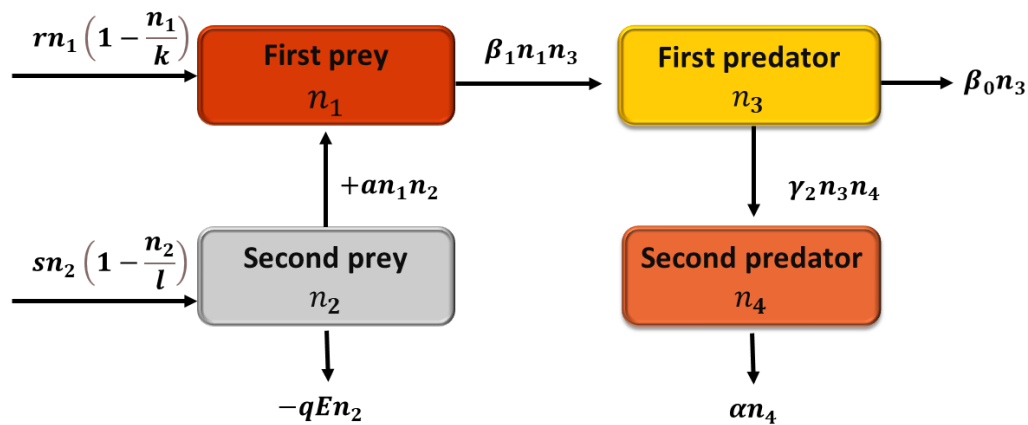


Figure 1. Block diagram for the model given by system (1).

3. Existence of Equilibria

The harvested food chain prey-predator model with a mutual interaction given by the system (1) has eight non-negative equilibrium points, namely:

1. $I_1 = (0,0,0,0)$, always exists.
2. $I_2 = (k, 0,0,0)$, always exists.
3. $I_3 = \left(0, \frac{l}{s}(s - qE), 0,0\right)$, exists when $s > qE$.
4. $I_4 = (\bar{n}_1, \bar{n}_2, 0,0)$, where $\bar{n}_1 = \frac{k}{r} [r + a\bar{n}_2]$ and $\bar{n}_2 = \frac{l}{s}(s - qE)$, exists when $s > qE$.
5. $I_5 = (\check{n}_1, 0, \check{n}_3, 0)$, where $\check{n}_1 = \frac{\beta_0}{\beta_2}$ and $\check{n}_3 = r \left(\frac{\beta_2 k - \beta_0}{\beta_1 \beta_2 k}\right)$, exists $\beta_2 k > \beta_0$.
6. $I_6 = (\hat{n}_1, \hat{n}_2, \hat{n}_3, 0)$, here $\hat{n}_1 = \frac{\beta_0}{\beta_2}$, $\hat{n}_2 = \frac{l}{s}(s - qE)$ and $\hat{n}_3 = \frac{rs(\beta_2 k - \beta_0) + alk\beta_2(s - qE)}{\beta_1 \beta_2 ks}$, which exists when $ralk\beta_2(s - qE) > s(\beta_0 - \beta_2 k)$.
7. $I_7 = (n_1, 0, n_3, n_4)$, here $n_1 = \frac{k}{r} \left(\frac{r\gamma_2 - \alpha\beta_1}{\gamma_2}\right)$, $n_3 = \frac{\alpha}{\gamma_2}$ and $n_4 = \frac{(r\gamma_2 - \alpha\beta_1)\beta_2 k - \beta_0 r\gamma_2}{r\gamma_1 \gamma_2}$, which exists when $k\beta_2(r\gamma_2 - \alpha\beta_1) > \beta_0 r\gamma_2$.

8. $I_8 = (n_1^*, n_2^*, n_3^*, n_4^*)$, $n_1^* = \frac{k[rs(\gamma_2 - \alpha\beta_1) + a\gamma_2(s - qE)]}{\gamma_2 sr}$, $n_2^* = \frac{L}{s}(s - qE)$, $n_3^* = \frac{\alpha}{\gamma_2}$ and $n_4^* = \frac{\beta_2 n_1^* - \beta_0}{\gamma_1}$, which exists when $a\gamma_2(s - qE) > rs(\alpha\beta_1 - \gamma_2)$ and $\beta_2 n_1^* > \beta_0$.

4. Local Bifurcation Analysis

This section studies the local bifurcation conditions near the equilibrium points of the system (1) using Sotomayor's theorem (Hubbard & West, 2013).

Now, define system (1) as:

$$\frac{dN}{dt} = F(N), \tag{2}$$

where, $N = (n_1, n_2, n_3, n_4)^T$ and $F = (n_1 f_1, n_2 f_2, n_3 f_3, n_4 f_4)$. $f_i: i = 1, 2, 3, 4$ represent the right-hand side functions of the system (1). The Jacobian matrix of the system (1) at each of the fixed points is given by:

$$J = \begin{bmatrix} r - \frac{2rn_1}{k} - \beta_1 n_3 + an_2 & an_1 & -\beta_1 n_1 & 0 \\ 0 & s - \frac{2sn_2}{l} - qE & 0 & 0 \\ \beta_2 n_3 & 0 & \beta_2 n_1 - \beta_0 - \gamma_1 n_4 & -\gamma_1 n_3 \\ 0 & 0 & \gamma_2 n_4 & \gamma_2 n_3 - \alpha \end{bmatrix}. \tag{3}$$

For nonzero vector $U = (u_1, u_2, u_3, u_4)^T$:

$$D^2F(U, U) = \begin{bmatrix} -2u_1 \left(\frac{r}{k} u_1 - au_2 + \beta_1 u_3 \right) \\ \frac{-2s}{l} u_2^2 \\ 2u_3(\beta_2 u_1 - \gamma_1 u_4) \\ 2\gamma_2 u_3 u_4 \end{bmatrix},$$

and, $D^3F(U, U, U) = (0, 0, 0, 0)^T$. So, according to Sotomayor theorem, the pitchfork bifurcation does not occur at each point $I_i, i = 1, 2, \dots, 8$.

The following theorem determines the saddle-node bifurcation of the system (1) at I_1 .

Theorem 1 For the parameter value $s^* = qE$, system (1), at the equilibrium point I_1 has a saddle-node bifurcation.

Proof According to the $J(I_1)$, system (1), at the equilibrium point I_1 , has a zero eigenvalue, say λ_{12} , at $s^* = qE$, and the Jacobian matrix $J^*(I_1) = J(I_1, s^*)$, becomes:

$$J^*(I_1) = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & s^* - qE & 0 & 0 \\ 0 & 0 & -\beta_0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Now, let $U^{[1]} = (u_1^{[1]}, u_2^{[1]}, u_3^{[1]}, u_4^{[1]})^T$ be the eigenvector corresponding to the eigenvalue

$\lambda_{12} = 0$. Thus $(J^*(I_1) - \lambda_{12}I)U^{[1]} = 0$, which implies: $u_1^{[1]} = u_3^{[1]} = u_4^{[1]} = 0$ and $u_2^{[1]}$ represents any nonzero real number. That means $U^{[1]} = (0, u_2^{[1]}, 0, 0)^T$.

Let $\Psi^{[1]} = (\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}, \psi_4^{[1]})^T$ be an eigenvector associated with the eigenvalue λ_{12} of the matrix J_1^{*T} . Then $(J_1^{*T} - \lambda_{12}I)\Psi^{[1]} = 0$. By solving this equation for $\Psi^{[1]}$, $\Psi^{[1]} = (0, \psi_2^{[1]}, 0, 0)^T$ is obtained, where $\psi_2^{[1]}$ is any nonzero real number.

Now, to check that the conditions of Sotomayor's theorem for transcritical bifurcation are satisfied, the following is measured:

$$\frac{\partial F}{\partial s} = F_s(N, s) = \left(\frac{\partial f_1}{\partial s}, \frac{\partial f_2}{\partial s}, \frac{\partial f_3}{\partial s}, \frac{\partial f_4}{\partial s} \right)^T = \left(0, \frac{l - n_2}{l}, 0, 0 \right)^T.$$

So, $F_s(I_1, s^*) = (0, 1, 0, 0)^T$ and hence $(\Psi^{[1]})^T F_s(I_1, s^*) = \psi_2^{[1]} \neq 0$. So transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$D^2 F_s(I_1, s^*)(U^{[1]}, U^{[1]}) = \left(0, -\frac{2qE [u_2^{[1]}]^2}{l}, 0, 0 \right)^T.$$

Hence,

$$\begin{aligned} (\Psi^{[1]})^T [D^2 F_s(I_1, s^*)(U^{[1]}, U^{[1]})] &= (0, \psi_2^{[1]}, 0, 0) \left(0, -\frac{2qE [u_2^{[1]}]^2}{l}, 0, 0 \right)^T \\ &= \frac{-2qE \psi_2^{[1]} [u_2^{[1]}]^2}{l} \neq 0. \end{aligned}$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_1 with the parameter $s^* = qE$.

The next theorem determines the saddle-node bifurcation of the system (1) at I_2 .

Theorem 2 For the parameter value $\beta_0^* = k\beta_2$, system (1), at the equilibrium point I_1 has a saddle-node bifurcation.

Proof According to $J(I_2)$, system (1), at I_2 , has a zero eigenvalue, say λ_{23} , at $\beta_0^* = k\beta_2$ and the Jacobian matrix $J^*(I_2) = J(I_2, \beta_0^*)$, becomes:

$$J^*(I_2) = \begin{bmatrix} -r & ak & -\beta_1 k & 0 \\ 0 & s - qE & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Now, let $U^{[2]} = (u_1^{[2]}, u_2^{[2]}, u_3^{[2]}, u_4^{[2]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{23} = 0$. Thus $(J^*(I_2) - \lambda_{23}I)U^{[2]} = 0$, which gives: $U^{[2]} = (u_1^{[2]}, 0, \frac{-r}{k\beta_1} u_1^{[2]}, 0)^T$, and $u_1^{[2]}$ is any nonzero real number.

Let $\Psi^{[2]} = (\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{23} = 0$ of the matrix J_2^{*T} . Then $(J_2^{*T} - \lambda_{23}I)\Psi^{[2]} = 0$. By solving this equation for $\Psi^{[2]}$, $\Psi^{[2]} =$

$(0,0, \Psi_3^{[2]}, 0)^T$ is obtained, where $\Psi_3^{[2]}$ is any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial \beta_0} = F_{\beta_0}(N, \beta_0) = \left(\frac{\partial f_1}{\partial \beta_0}, \frac{\partial f_2}{\partial \beta_0}, \frac{\partial f_3}{\partial \beta_0}, \frac{\partial f_4}{\partial \beta_0} \right)^T = (0, 0, -1, 0)^T.$$

So, $F_{\beta_0}(I_2, \beta_0^*) = (0, 0, -1, 0)^T$ and hence $(\Psi^{[2]})^T F_{\beta_0}(I_2, \beta_0^*) = \Psi_3^{[2]} \neq 0$. Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$D^2 F_{\beta_0}(I_2, \beta_0^*)(U^{[2]}, U^{[2]}) = \left(0, 0, \frac{-2r\beta_2 [u_1^{[2]}]^2}{\beta_1}, 0 \right)^T,$$

hence, it is obtained that:

$$(\Psi^{[2]})^T [D^2 F_{\beta_0}(I_2, \beta_0^*)(U^{[2]}, U^{[2]})] = \frac{-2r\beta_2 \Psi_3^{[2]} [u_1^{[2]}]^2}{\beta_1} \neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_2 with the parameter $\beta_0^* = k\beta_2$.

The next theorem determines the saddle-node bifurcation of the system (1) at the equilibrium point I_3 .

Theorem 3 For the parameter value $\beta_0^* = k\beta_2$, system (1), at the equilibrium point I_3 has a saddle-node bifurcation.

Proof According to $J(I_3)$, system (1), at the equilibrium point I_3 , has a zero eigenvalue, say λ_{32} , at $s^\# = qE$, and the Jacobian matrix $J_3^\# = J(I_3, s^\#)$, becomes:

$$J_3^\# = \begin{bmatrix} r + a\dot{n}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Now, let $U^{[3]} = (u_1^{[3]}, u_2^{[3]}, u_3^{[3]}, u_4^{[3]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{32} = 0$. Thus $(J_3^\# - \lambda_{32}I)U^{[3]} = 0$, which gives: $U^{[3]} = (0, u_2^{[3]}, 0, 0)^T$, and $u_2^{[3]}$ is any nonzero real number.

Let $\Psi^{[3]} = (\Psi_1^{[3]}, \Psi_2^{[3]}, \Psi_3^{[3]}, \Psi_4^{[3]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{32} = 0$ of the matrix $J_3^{\#T}$. Then, $(J_3^{\#T} - \lambda_{32}I)\Psi^{[3]} = 0$. By solving this equation for $\Psi^{[3]}$, $\Psi^{[3]} = (0, \Psi_2^{[3]}, 0, 0)^T$ is obtained, where $\Psi_2^{[3]}$ is any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial s} = F_s(N, s) = \left(\frac{\partial f_1}{\partial s}, \frac{\partial f_2}{\partial s}, \frac{\partial f_3}{\partial s}, \frac{\partial f_4}{\partial s} \right)^T = \left(0, \frac{l-n_2}{l}, 0, 0 \right)^T.$$

So, $F_s(I_3, s^\#) = (0, 1, 0, 0)^T$ and hence $(\Psi^{[3]})^T F_s(I_3, s^\#) = \Psi_2^{[3]} \neq 0$.

Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$D^2 F_s(I_3, s^\#)(U^{[3]}, U^{[3]}) = \left(0, -\frac{2qE [u_2^{[3]}]^2}{l}, 0, 0 \right)^T$$

Hence, it is obtained that:

$$(\Psi^{[3]})^T [D^2 F_s(I_3, s^\#)(U^{[3]}, U^{[3]})] = -\frac{2qE \Psi_2^{[3]} [u_2^{[3]}]^2}{l} \neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_3 with the parameter $s^\# = qE$.

The next theorem determines the saddle-node bifurcation of the system (1) at the equilibrium point I_4 .

Theorem 4 For the parameter value $\beta_2^\# = \frac{\beta_0}{\bar{n}_1}$, system (1), at the equilibrium point I_4 has a saddle-node bifurcation.

Proof According to the $J(I_4)$ system (1), at the equilibrium point I_4 , has a zero eigenvalue, say λ_{23} , at $\beta_2^\# = \frac{\beta_0}{\bar{n}_1}$. And the Jacobian matrix $J_4^\# = J(I_4, \beta_2^\#)$, becomes:

$$J_4^\# = \begin{bmatrix} -(r + a\bar{n}_2) & a\bar{n}_1 & -\beta_1\bar{n}_1 & 0 \\ 0 & -(s - qE) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Now, let $U^{[4]} = (u_1^{[4]}, u_2^{[4]}, u_3^{[4]}, u_4^{[4]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{43} = 0$. Thus $(J_4^\# - \lambda_{43}I)V^{[4]} = 0$, which gives: $U^{[4]} = (u_1^{[4]}, 0, \frac{-(r+a\bar{n}_2)}{\beta_1\bar{n}_1} u_1^{[4]}, 0)^T$, and $u_1^{[4]}$ is any nonzero real number.

Let $\Psi^{[4]} = (\Psi_1^{[4]}, \Psi_2^{[4]}, \Psi_3^{[4]}, \Psi_4^{[4]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{43} = 0$ of the matrix $J_4^{\#T}$.

Then, $(J_4^{\#T} - \lambda_{43}I)\Psi^{[4]} = 0$. By solving this equation for $\Psi^{[4]}$, $\Psi^{[4]} = (0, 0, \Psi_3^{[4]}, 0)^T$ is obtained, where $\Psi_3^{[4]}$ is any nonzero real number.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial \beta_2} = F_{\beta_2}(N, \beta_2) = \left(\frac{\partial f_1}{\partial \beta_2}, \frac{\partial f_2}{\partial \beta_2}, \frac{\partial f_3}{\partial \beta_2}, \frac{\partial f_4}{\partial \beta_2} \right)^T = (0, 0, n_1, 0)^T.$$

So, $F_{\beta_2}(I_4, \beta_2^\#) = (0, 0, \bar{n}_1, 0)^T$ and hence $(\Psi^{[4]})^T F_{\beta_2}(I_4, \beta_2^\#) = \bar{n}_1 \Psi_3^{[4]} \neq 0$. Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$(\Psi^{[4]})^T [D^2 F_{\beta_2}(I_4, \beta_2^\#)(U^{[4]}, U^{[4]})] = \frac{2\beta_0}{\bar{n}_1} u_1^{[4]} u_3^{[4]} \Psi_3^{[4]} \neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_4 with the parameter $\beta_2^\# = \frac{\beta_0}{\bar{n}_1}$.

The next theorem determines the saddle-node bifurcation of the system (1) at the equilibrium point I_5 .

Theorem 5 For the parameter value $\beta_2^\# = \frac{\beta_0}{\bar{n}_1}$, system (1), at the equilibrium point I_5 has a saddle-node bifurcation.

Proof According to $J(I_5)$, system (1), at the equilibrium point I_5 , has a zero eigenvalue, say λ_{54} , at $\gamma_2^* = \frac{\alpha}{\bar{n}_3}$. And the Jacobian matrix $J_5^* = J(I_5, \gamma_2^*)$, becomes:

$$J_5^* = \begin{bmatrix} -r\beta_0 & a\beta_0 & -\beta_0\beta_1 & 0 \\ k\beta_2 & \beta_2 & \beta_2 & 0 \\ 0 & s - qE & 0 & 0 \\ \beta_2\ddot{n}_3 & 0 & 0 & -\gamma_1\ddot{n}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, let $U^{[5]} = (u_1^{[5]}, u_2^{[5]}, u_3^{[5]}, u_4^{[5]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{54} = 0$. Thus $(J_5^* - \lambda_{54}I)V^{[5]} = 0$, which gives: $U^{[5]} = (u_1^{[5]}, 0, \frac{-r}{k\beta_1}u_1^{[5]}, \frac{\beta_2}{\gamma_1}u_1^{[5]})^T$, where $u_1^{[5]}$ is any nonzero real number. Let $\Psi^{[5]} = (\psi_1^{[5]}, \psi_2^{[5]}, \psi_3^{[5]}, \psi_4^{[5]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{54} = 0$ of the matrix J_5^{*T} . Then $(J_5^{*T} - \lambda_{54}I)\Psi^{[5]} = 0$. By solving this equation for $\Psi^{[5]}$, $\Psi^{[5]} = (0, 0, 0, \psi_4^{[5]})^T$ is obtained, where $\psi_4^{[5]}$ is any nonzero real number.

Now, consider:

$$\frac{\partial F}{\partial \gamma_2} = F_{\gamma_2}(N, \gamma_2) = \left(\frac{\partial f_1}{\partial \gamma_2}, \frac{\partial f_2}{\partial \gamma_2}, \frac{\partial f_3}{\partial \gamma_2}, \frac{\partial f_4}{\partial \gamma_2}\right)^T = (0, 0, 0, n_3)^T.$$

So, $F_{\gamma_2}(I_5, \gamma_2^*) = (0, 0, 0, \ddot{n}_3)^T$ and hence $(\Psi^{[5]})^T F_{\gamma_2}(I_5, \gamma_2^*) = \ddot{n}_3 \psi_4^{[5]} \neq 0$. Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$(\Psi^{[5]})^T [D^2 F_{\gamma_2}(I_5, \gamma_2^*)(U^{[5]}, U^{[5]})] = 2\gamma_2^* u_3^{[5]} u_4^{[5]} \Psi_4^{[5]} \neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_5 with the parameter $\gamma_2^* = \frac{\alpha}{\bar{n}_3}$.

The next theorem determines the saddle-node bifurcation of the system (1) at the equilibrium point I_6 .

Theorem 6 For the parameter value $\beta_2^\# = \frac{\beta_0}{\hat{n}_1}$, system (1), at the equilibrium point I_6 has a saddle-node bifurcation.

Proof According to $J(I_6)$, system (1), at the equilibrium point I_6 , has a zero eigenvalue, say λ_{64} , at $\gamma_2^\# = \frac{\alpha}{\hat{n}_3}$. And the Jacobian matrix $J_6^* = J(I_6, \gamma_2^\#)$, becomes:

$$J_6^\# = \begin{bmatrix} -\frac{r\hat{n}_1}{k} & a\hat{n}_1 & -\beta_1\hat{n}_1 & 0 \\ 0 & -(s - qE) & 0 & 0 \\ \beta_2\hat{n}_3 & 0 & 0 & -\gamma_1\hat{n}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, let $U^{[6]} = (u_1^{[6]}, u_2^{[6]}, u_3^{[6]}, u_4^{[6]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{64} = 0$. Thus $(J_6^\# - \lambda_{64}I)V^{[6]} = 0$, which gives: $U^{[6]} = (u_1^{[6]}, 0, \frac{-r}{k\beta_1}u_1^{[6]}, \frac{\beta_2}{\gamma_1}u_1^{[6]})^T$, where $u_1^{[6]}$ is any nonzero real number.

Let $\Psi^{[6]} = (\psi_1^{[6]}, \psi_2^{[6]}, \psi_3^{[6]}, \psi_4^{[6]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{64} = 0$ of the matrix $J_6^{\#T}$. Then $(J_6^{\#T} - \lambda_{64}I)\Psi^{[6]} = 0$. By solving this equation for $\Psi^{[6]}$, $\Psi^{[6]} = (0, 0, 0, \psi_4^{[6]})^T$ is obtained, where $\psi_4^{[6]}$ is any nonzero real number. Now, consider:

$$\frac{\partial F}{\partial \gamma_2} = F_{\gamma_2}(N, \gamma_2) = \left(\frac{\partial f_1}{\partial \gamma_2}, \frac{\partial f_2}{\partial \gamma_2}, \frac{\partial f_3}{\partial \gamma_2}, \frac{\partial f_4}{\partial \gamma_2} \right)^T = (0, 0, 0, n_3)^T.$$

So, $F_{\gamma_2}(I_6, \gamma_2^\#) = (0, 0, 0, \hat{n}_3)^T$ and hence $(\Psi^{[6]})^T F_{\gamma_2}(I_6, \gamma_2^\#) = \hat{n}_3 \psi_4^{[6]} \neq 0$. Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$(\Psi^{[6]})^T [D^2 F_{\gamma_2}(I_6, \gamma_2^\#)(U^{[6]}, U^{[6]})] = 2\gamma_2^\# u_3^{[6]} u_4^{[6]} \psi_4^{[6]} \neq 0.$$

Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_6 with the parameter $\gamma_2^\# = \frac{\alpha}{\hat{n}_3}$.

The next theorem determines the saddle-node bifurcation of the system (1) at the equilibrium point I_7 .

Theorem 7 For the parameter value $\beta_2^\# = \frac{\beta_0}{\hat{n}_1}$, system (1), at the equilibrium point I_7 has a saddle-node bifurcation.

Proof According to $J(I_7)$, system (3.1), at the equilibrium point I_7 , has a zero eigenvalue, say λ_{72} , at $\tilde{S} = qE$, and the Jacobian matrix $\tilde{J}_7 = J(I_7, \tilde{S})$, becomes:

$$\tilde{j}_7 = \begin{bmatrix} -(r - \beta_1 \tilde{n}_3) & a\tilde{n}_1 & -\beta_1 \tilde{n}_1 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_2 \tilde{n}_3 & 0 & 0 & -\gamma_1 \tilde{n}_3 \\ 0 & 0 & \gamma_2 \tilde{n}_4 & 0 \end{bmatrix}.$$

Now, let $U^{[7]} = (u_1^{[7]}, u_2^{[7]}, u_3^{[7]}, u_4^{[7]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{72} = 0$. Thus $(\tilde{j}_7 - \lambda_{72}I)U^{[7]} = 0$, which gives: $U^{[7]} = (\frac{\gamma_1}{\beta_2}u_4^{[7]}, \frac{(r - \beta_1 \tilde{n}_3)\gamma_1}{a\tilde{n}_1\beta_2}u_4^{[7]}, 0, u_4^{[7]})^T$, where $u_4^{[7]}$ is any nonzero real number. Let $\Psi^{[7]} = (\psi_1^{[7]}, \psi_2^{[7]}, \psi_3^{[7]}, \psi_4^{[7]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{72} = 0$ of the matrix \tilde{j}_7^T . Then $(\tilde{j}_7^T - \lambda_{72}I)\Psi^{[7]} = 0$. By solving this equation for $\Psi^{[7]}$: $\Psi^{[7]} = (0, \psi_2^{[7]}, 0, 0)^T$ is obtained, where $\psi_2^{[7]}$ is any nonzero real number.

Now, consider:

$$\frac{\partial F}{\partial s} = f_s(N, s) = \left(\frac{\partial f_1}{\partial s}, \frac{\partial f_2}{\partial s}, \frac{\partial f_3}{\partial s}, \frac{\partial f_4}{\partial s}\right)^T = \left(0, \frac{l-n_2}{l}, 0, 0\right)^T.$$

Then, $F_s(I_7, \tilde{S}) = (0, 1, 0, 0)^T$ and hence $(\Psi^{[7]})^T F_s(I_7, \tilde{S}) = \psi_2^{[7]} \neq 0$. Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met.

Now,

$$(\Psi^{[7]})^T [D^2 f_s(I_7, \tilde{S})(U^{[7]}, U^{[7]})] = \frac{-2qE}{l} \psi_2^{[7]} \neq 0.$$

Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_6 with the parameter $\tilde{S} = qE$.

The next theorem determines the saddle-node bifurcation of the system (1) at the equilibrium point I_8 .

Theorem 8 For the parameter value $s^* = qE$, system (1), at the equilibrium point I_8 has a saddle-node bifurcation.

Proof According to $J(I_8)$, system (1), at the equilibrium point I_8 , has a zero eigenvalue, say λ_{82} , at $s^* = qE$, and the Jacobian matrix $J_8^* = J(I_8, s^*)$, becomes:

$$J_8^* = \begin{bmatrix} r - \frac{2rn_1^*}{k} - \beta_1 n_3^* + an_2^* & an_1^* & -\beta_1 n_1^* & 0 \\ 0 & 0 & 0 & 0 \\ \beta_2 n_3^* & 0 & 0 & -\gamma_1 n_3^* \\ 0 & 0 & \gamma_2 n_4^* & 0 \end{bmatrix}.$$

Now, let $U^{[8]} = (u_1^{[8]}, u_2^{[8]}, u_3^{[8]}, u_4^{[8]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{82} =$

0. Thus $(J_8^* - \lambda_{82}I)U^{[8]} = 0$, which gives: $U^{[8]} = \left(u_1^{[8]}, \frac{-(r - \frac{2rn_1^*}{k} - \beta_1 n_3^* + an_2^*)}{an_1^*} u_1^{[8]}, 0, \frac{\beta_2}{\gamma_1} u_1^{[8]} \right)^T$,

where $u_1^{[8]}$ is any nonzero real number. Let $\Psi^{[8]} = (\Psi_1^{[8]}, \Psi_2^{[8]}, \Psi_3^{[8]}, \Psi_4^{[8]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{82} = 0$ of the matrix J_8^{*T} . Then $(J_8^{*T} - \lambda_{82}I)\Psi^{[8]} = 0$. By solving this equation for $\Psi^{[8]}$, $\Psi^{[8]} = (0, \Psi_2^{[8]}, 0, 0)^T$ is obtained, where $\Psi_2^{[8]}$ is any nonzero real number.

Now,

$$\frac{\partial F}{\partial s} = f_s(N, s) = \left(\frac{\partial f_1}{\partial s}, \frac{\partial f_2}{\partial s}, \frac{\partial f_3}{\partial s}, \frac{\partial f_4}{\partial s} \right)^T = \left(0, \frac{l - n_2}{l}, 0, 0 \right)^T$$

So, $F_s(I_8, s^*) = (0, 1, 0, 0)^T$ and hence $(\Psi^{[8]})^T F_s(I_8, s^*) = \Psi_2^{[8]} \neq 0$.

Therefore, transcritical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$(\Psi^{[8]})^T [D^2 f_s(I_8, s^*)(U^{[8]}, U^{[8]})] = \frac{-2qE}{l} [u_2^{[8]}]^2 \Psi_2^{[8]} \neq 0.$$

Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_8 with the parameter $s^* = qE$.

5. Numerical Simulation and Discussion

This section aims to find the system's critical parameters that affect the behaviour of the proposed system by using numerical simulations. The dynamics of system (1) are obtained by solving system (1) numerically and then drawing the time series of the solutions of system (1) for different sets of parameters. Now, for the following set of parameters:

$$r = 1, k = 5, a = 0.4, \beta_1 = 2, a = 0.4, \alpha = 0.2, qE = 0.4, s = 0.9, l = 4, \beta_2 = 1.25, \gamma_1 = 0.6, \gamma_2 = 0.54, \beta_0 = 1, \tag{4}$$

The equilibrium point I_8 exists, and it is given by $(n_1^*, n_2^*, n_3^*, n_4^*) = (3.88, 2.22, 0.74, 4.76)$.

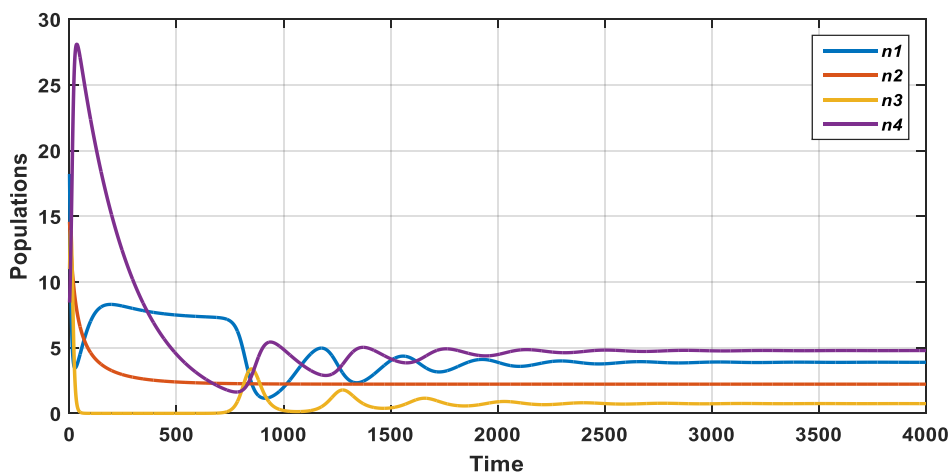


Figure 2. Dynamics of the four species with the data given by Eq. (4).

Figure 3 explains the system's dynamics with the data given by Eq. (4) with different values of a (the positive effect rate on the first prey by the second prey). It illustrates the solution of system (1) settling down to I_8 for different values of a . This means that all species survive for a long time, and therefore, no bifurcation might occur.

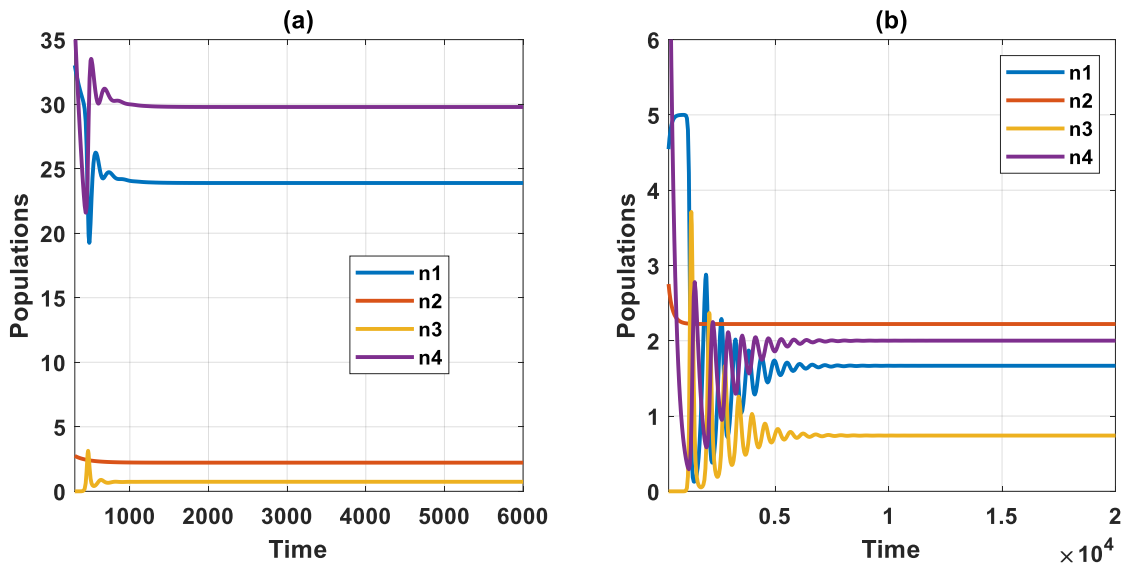


Figure 3. Dynamics of the four species with the data given by Eq. (4) with (a) $a=2$, system (1) converges to $(23.8, 2.2, 0.7, 29.7)$ (b) $a=0.0001$, system (1) converges to $(1.6, 2.2, 0.7, 2)$.

Figure 4 studies the dynamics of the system (1) with the data given by Eq. (4) with different values of β_1 (the attack rate coefficient of the first prey due to the first predator). It shows the second predator becomes zero when $\beta_1 \geq 2.1$. Furthermore, the first predator faces extinction when $\beta_1 \leq 0.01$. On the other hand, the solution of the system (1) approaches to I_8 when $0.01 < \beta_1 < 2.1$. This shows that the bifurcation occurred when $\beta_1 \in \{0.01, 2.1\}$.

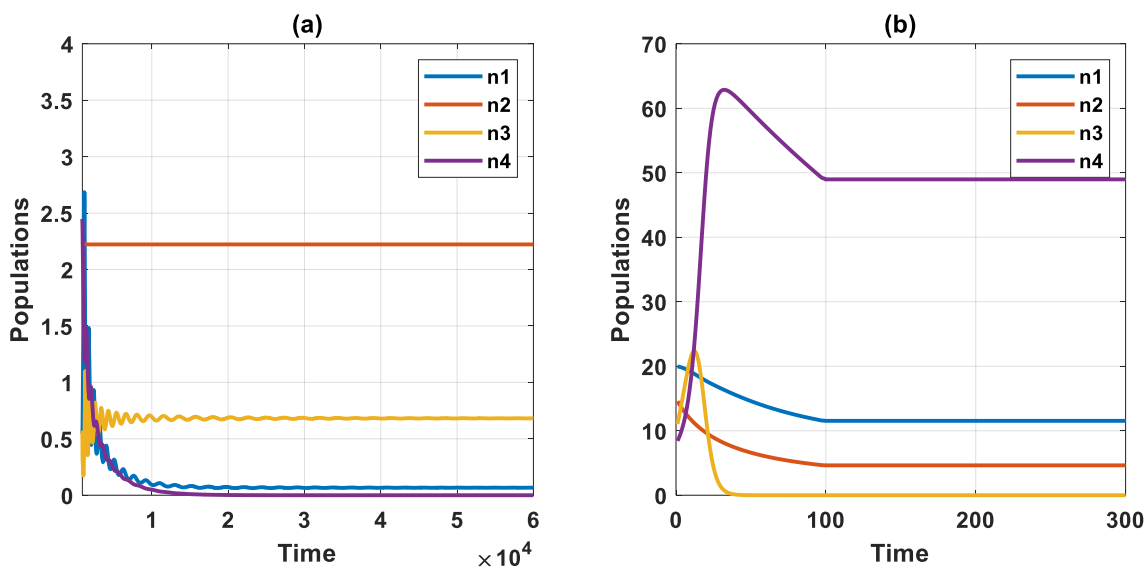


Figure 4. Dynamics of the four species with the data given by Eq. (4) with (a) $\beta_1=2.1$, system (1) converges to $(0.06, 2.2, 0.6, 0)$. (b) $\beta_1=0.01$, system (1) converges to $(11.5, 4.6, 0, 48.9)$.

Figure 5 illustrates the dynamics of the system (1) with the data given by Eq. (3) with different values of β_2 . It shows the first predator becomes zero when $\beta_2 \geq 1.97$. Moreover, the second predator faces extinction when $\beta_2 \leq 0.026$. On the other hand, the solution of the system (1) approaches to I_8

when $0.026 < \beta_2 < 1.97$ that leads to the occurrence of local bifurcation when $\beta_2 \in \{0.026, 1.97\}$.

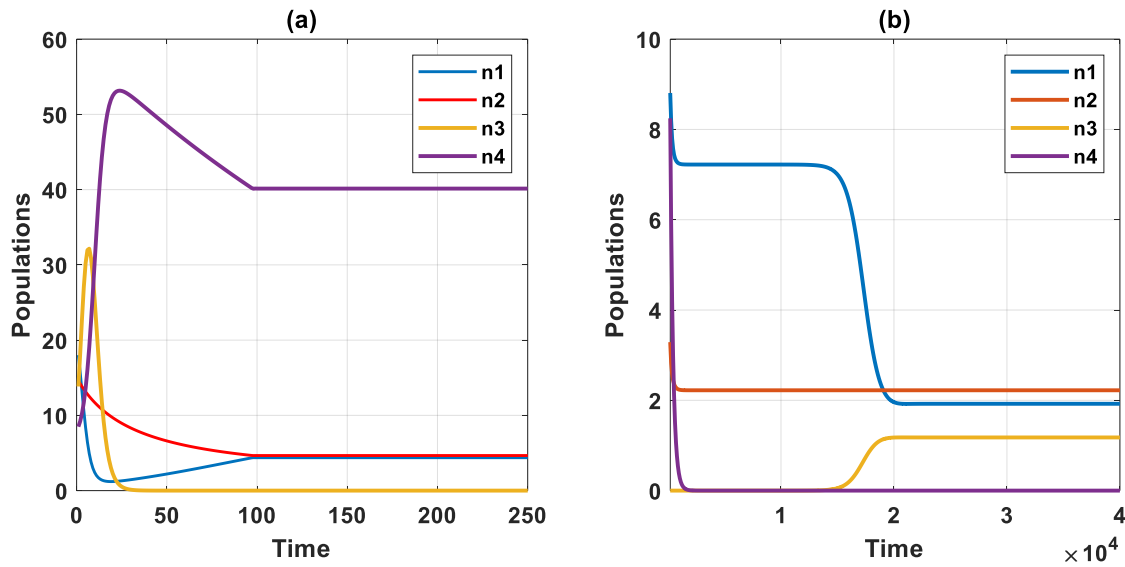


Figure 5. Dynamics of the four species with the data given by Eq. (4) with (a) $\beta_2=1.97$, system (1) converges to (4.3, 4.6, 0, 40). (b) $\beta_2=0.026$, system (1) converges to (1.9, 2.2, 1.1, 0).

Figure 6 explains the system's dynamics (1) with the data given by Eq. (4) with different values of qE (harvesting rate). It illustrates the solution of system (1) settling down to I_7 in the Int. $R^3_{+(n_1 n_3 n_4)}$ when $qE \geq 0.9$. While all species keep surviving for $qE < 0.9$. That means system (1) faces a bifurcation at $qE = 0.9$.

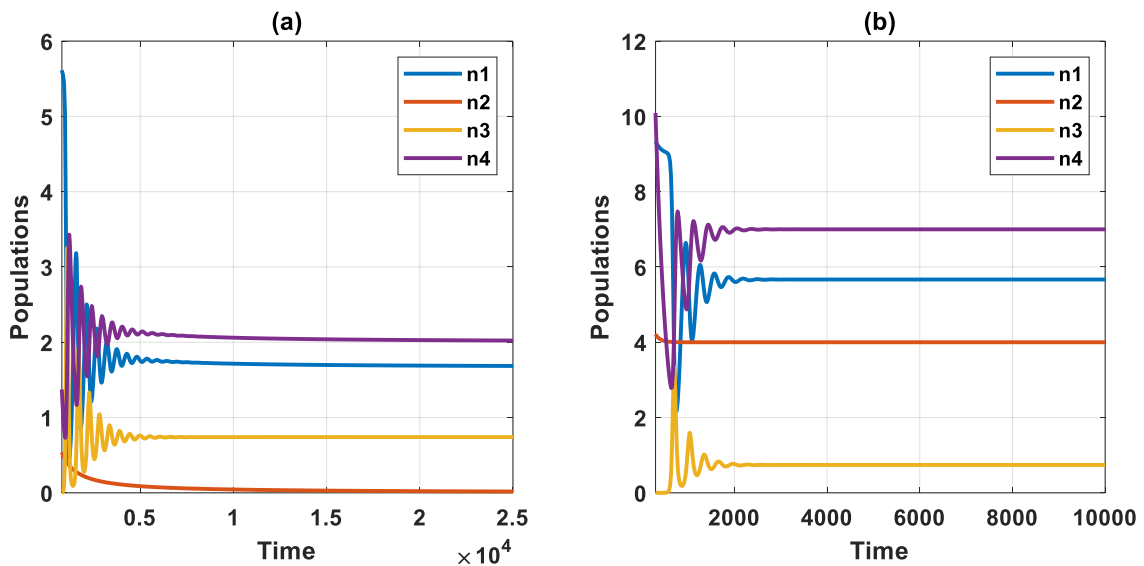


Figure 6. Dynamics of the four species with the data given by Eq. (4) with (a) $qE=0.9$, system (1) converges to (1.6, 0, 0.7, 2). (b) $qE=0.00004$, system (1) converges to (5.6, 3.9, 0.7, 6.9).

Figure 7 explains the system's dynamics with the data given by Eq. (3) with different values of s (intrinsic growth rate of second prey). It illustrates the solution of system (1) settling down to I_7 in the Int. $R^3_{+(n_1 n_3 n_4)}$ when $s \leq 0.39$. While the system (1) keep persists for $s > 0.39$. This shows that the bifurcation occurred when $s = 0.39$.

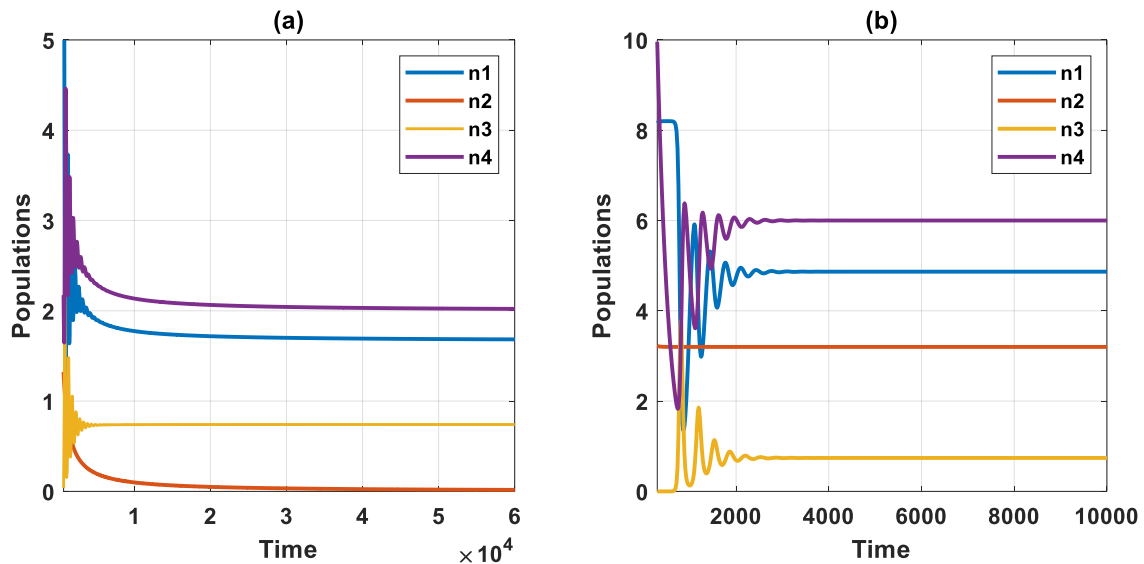


Figure 7. Dynamics of the four species with the data given by Eq. (4) with (a) $s=0.39$, system (1) converges to $(1.6, 0, 0.7, 2)$. (b) $s=2$, system (1) converges to $(2.8, 3.2, 0.7, 6)$.

6. Discussion

This study proposes one commensalism prey, one harvested prey, predator and super predator model. The goal was to comprehend how this kind of interaction affected the prey–predator system dynamics. The system experienced theoretical and numerical analysis. The system was shown to have eight equilibrium points. These equilibrium points show local stability under certain conditions.

The system has a stable point attractor that may transfer to being unstable at the bifurcation point. Further, the local stability conditions have been violated at the bifurcation points. However, the numerical simulation achieved for the chosen hypothetical data set revealed a rich dynamical behaviour that may be summed up in the next section.

7. Conclusion

A commensalism ecological model, which describes predation and harvesting on the dynamical behaviour of a food chain prey-predator model with a Lotka-Volterra type of functional response, has been suggested and studied. The mathematical analysis has shown that system (1) has eight non-negative equilibrium points. The conditions that guarantee the occurrence of local bifurcation of system (1) around each equilibrium point have been introduced. System (1) has been solved numerically to approve the analytical results. The impact of various parameters on the dynamic behaviour of the given system has been investigated, with the following results being found.

1. The first prey species n_1 is survived under all conditions.
2. It is detected that the system keep persists if the parameter a is varied.
3. The most critical parameter that affects the system's behaviour is $s - qE$ (the relation between the intrinsic growth rates s and the harvesting rate qE). The paper shows that most of the equilibrium points' existence and stability relies on the relationship between these two parameters.

Overall, the system with a positive effect on the first prey and harvesting restriction stabilizes the ecosystem.

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Conflicts of Interest

The authors declare no conflict of interest.

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