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ARTICLE

A New Model of Mixture Distribution Using a Survival Analysis of Cancer Patients[1]

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Abstract
In this article, specific statistical considerations are typically required in order to select the best model for fitting cancer survival data. A new two-parameter distribution known as the New Mixture of Lomax and Gamma Distribution (MLGD) is proposed in this article. Because of the unique way that the gamma and Lomax distributions are mixed, this distribution is created as a special mixture of two distributions. Statistical properties, order statistics, entropy, and reliability analysis are also derived. The maximum likelihood estimation method can be used to estimate the parameters of the distribution. Lastly, a goodness-of-fit analysis is demonstrated on a set of data on cancer survival. It is compared to the fit and shows that the new Lomax and gamma mixing distributions have more flexibility than the other distributions.
Keywords: Mixture distribution; Reliability analysis; Moments; Order Statistics; Maximum likelihood Estimation.

1. Introduction

Medical research is mostly interested in studying the survival of cancer patients, as applied to statistical distributions. The statistical distributions have been extensively utilized for analyzing time-to-event data, also referred to as survival or reliability data, in different areas of applicability, including medical science. In recent years, an impressive set of new statistical distributions has been explored by statisticians. The necessity of developing an extended class of classical distribution is analysis, biomedicine, reliability, insurance, and finance. Recently, many researchers have been working in this area and have proposed new methods to develop improved probability distributions with utility. A statistical study is frequently used, which extensively depends on the presumptive probability model or distributions.

A Lomax *et al.* (1954) distribution RV X with a parameter $\alpha,\beta > 0$ is described by its pdf, which is defined as

$$f(x,\alpha,\beta) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \quad ,x > 0,\alpha,\beta > 0$$

Considering the gamma [17] distribution with parameters $\alpha = 3, and\ \beta$ the pdf can be defined as

$$f(x;\alpha) = \frac{1}{2}\alpha^3 x^2\, e^{-\alpha x} \quad ,x > 0,\alpha > 0$$

The concept of a finite mixture of probabilities was pioneered by Newcomb (1886) as a model for outliers. Weldon (1892) provided a mixture technique for analyzing crab morphometric data. Pearson (1894) introduced a statistical model using finite mixtures of normal distributions and also estimated the parameters of the mixture. Fisher (1934) introduced the concept of a weighted mixture of outcomes and developed the Sib method. He applied this method for analyzing medical, biological, and agricultural sciences data with randomly biassed samples. Teichroew (1957) considered the mixture of normal distributions with different variances and derived the marginal distribution when the variances are assumed to follow the gamma distribution. He also obtained the properties of the new distribution. Lindley (1958) introduced the fiducial distribution and Bayes theorem. Rama Shanker (2015), has introduced a mixture of exponential (θ) and gamma (2, θ) distribution proposed a shanker distribution. Akash distribution is a two-component mixture of an exponential distribution and gamma distribution with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$, Shanker (2015). Proposed a Komal distribution with applications in survival analysis, Ramma Shanker (2023), the combination of exponential (θ) and gamma (2, θ) distribution with mixing proportions $\frac{\theta(\theta+1)}{\theta^2+\theta+1}$ and $\frac{1}{\theta^2+\theta+1}$. Recently, Telee *et al*. (2024), introduced the modified Lomax distribution properties and applications. Tahir *et al*. (2015), The Weibull Lomax distribution properties and applications. Abiodun *et al*. (2022), introduced the On Maxwell–Lomax distribution properties and applications.

This article is based on a new mixture of Lomax and Gamma distributions in order to create the MLGD that was proposed. For the remainder of this research work, the presented (pdf) and (cdf) functions of the proposed distribution, together with some of its properties, provide an approach to the maximum likelihood estimators for estimating the model parameters. Finally, the results of fitting the caner survival data with MLGD also show that the other well-known distributions. Throughout this research, the statistical programming language R was used for all computations.

2. New Mixture of MLGD Distribution

In this section introduces the MLGD distribution, which is a new distribution created by combining two existing distributions. Let X be a random variable with a mixed distribution. Its density function (pdf) $f(x)$, is expressed as follows:

$$f(x) = \sum_{i=1}^{k} \omega_i f_i(x)$$

$f_i(x), i = 1, 2, \dots, k$ probability density function for all i

$\omega_i,\ i = 1, \dots, k$ denote mixing proportions that are non-negative and $\sum_{i=1}^{k} \omega_i = 1$.

The $f_1(x) \sim gamma\ (\alpha = 3, \beta\ )$ and $f_2(x) \sim Lomax\ (\alpha, \beta)$ two independents random variables with $\frac{\beta}{\beta+1}\, and\ \frac{1}{\beta+1}$ respectively. Then, the density function of the mixed distribution X is given by.

$$f(x;\ \alpha, \beta) = \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\, e^{-\alpha x}\right), x > 0, \alpha > 0, \beta > 0 \qquad (2.1)$$

The function defined in Eq. (2.1) represents a probability density function pdf of the *new mixture of Lomax and Gamma distributions, denoted as the (MLGD)* $f(x; \alpha, \beta), for\ all\ x > 0$

$$f(x; \alpha, \beta) = \int_0^{\infty} \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\, e^{-\alpha x}\right)dx \qquad (2.2)$$

Integration, using the substitution method in Eq. (2.2)

$$u = 1 + \frac{x}{\beta}, du = \frac{dx}{\beta}, then\ \ dx = du\beta$$

Let’s, make a substitution to simplify the integral in Eq. (2.2)

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta+1}\left(\frac{1}{\alpha}+\frac{\beta}{\alpha}\right)$$

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta+1}\left(\frac{\beta+1}{\alpha}\right)$$

$$f(x;\alpha,\beta)=1$$

The cumulative distribution function cdf of new mixture of Lomax and Gamma distributions is defined as

$$F(x;\ \alpha,\beta)=\frac{\alpha}{\beta+1}\int_0^x\left(\frac{1}{\beta}\left(1+\frac{z}{\beta}\right)^{-(\alpha+1)}+\frac{\alpha^2\beta}{2}z^2\,e^{-\alpha z}\right)dz\quad x\geq 0,\alpha,\beta>0 \tag{2.3}$$

First, term integration

$$F(x;\ \alpha,\beta)=\int_0^x\frac{1}{\beta}\left(1+\frac{z}{\beta}\right)^{-(\alpha+1)}\ dz \tag{2.4}$$

Integration using the substitution method

Let's assuming,

$$u=1+\frac{z}{\beta}\ then,\qquad du=\frac{dz}{\beta},or\ \ dz=du\beta$$

$$Then,\qquad z=0,\qquad u=1\ and\ \ z=x,\qquad u=1+\frac{x}{\beta}$$

Substituting this back into the integral in Eq. (2.4), we have

$$F(x;\ \alpha,\beta)=\frac{\alpha}{\beta+1}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\alpha}\right) \tag{2.5}$$

Second term integration

$$F(x;\ \alpha,\beta)=\frac{\alpha^2\beta}{2}\int_0^x z^2\,e^{-\theta z}\,dz \tag{2.6}$$

Using, the integration by substitution method in Eq. (2.6)

By letting, $u=z^2, and\ dv=e^{-\theta z}$,

Simplifying this, we get

$$F(x;\ \alpha,\beta)=\frac{\beta}{\beta+1}\left(1-\left(\left(\frac{\alpha^2x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}\right) \tag{2.7}$$

Let's add these integrals together in Eqs. (2.5) and (2.7).

$$F(x;\ \alpha,\beta)=\frac{1}{\beta+1}\left((\beta+1)-\left(\left(1+\frac{x}{\beta}\right)^{-\alpha}+\beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}\right)\right)$$

$$F(x;\ \alpha,\beta)=1-\left(\frac{\left(1+\frac{x}{\beta}\right)^{-\alpha}+\beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}{\beta+1}\right) \tag{2.8}$$

Then, using the following binomial series expansion in Eq. (2.8)

$$(1+x)^n=\sum_{j=0}^{\infty}\binom{n}{j}x^j \tag{2.9}$$

Then, the cumulative distribution function cdf of the *new mixture of Lomax and Gamma distributions*, *denoted as the (MLGD)* are obtained as

$$F(x;\ \alpha,\beta) = 1 - \left(\frac{\sum_{j=0}^{\infty}(-1)^j \binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j + \beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}{\beta+1}\right), x \geq 0, \alpha, \beta > 0 \quad (2.10)$$

3. Reliability Analysis

This section will provide the reliability function, hazard function, reverse hazard function, cumulative hazard function, odds rate, and mean residual function for the specified MLG distribution.

3.1 Survival Function

The survival function of the MLG distribution is defined as
$S(x;\alpha,\beta) = 1 - F(x;\alpha,\beta)$

$$S(x;\alpha,\beta) = \left(\frac{\sum_{j=0}^{\infty}(-1)^j \binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j + \beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}{\beta+1}\right) \quad (3.1)$$

3.2 Hazard Rate Function

An important metric for describing life phenomena is the hazard rate function of the MLG distribution, which is defined by. $h(x) = \frac{f(x;\alpha,\beta)}{1-F(x;\alpha,\beta)}$

$$h(x;\alpha,\beta) = \left(\frac{\alpha\left(\frac{1}{\alpha}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\, e^{-\alpha x}\right)}{\left(\frac{\sum_{j=0}^{\infty}(-1)^j \binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j + \beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}{\beta+1}\right)}\right) \quad (3.2)$$

3.3 Revers hazard rate

The Revers hazard rate of the MLG distribution is defined as

$$h_r(x;\alpha,\beta) = \frac{f(x;\alpha,\beta)}{F(x;\alpha,\beta)}$$

$$h_r(x;\alpha,\beta) = \left(\frac{\alpha\left(\frac{1}{\alpha}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\, e^{-\alpha x}\right)}{(\beta+1) - \left(\sum_{j=0}^{\infty}(-1)^j \binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j + \beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}\right)}\right) \quad (3.3)$$

3.4 Cumulative hazard function

The Cumulative hazard function of the MLG distribution is defined as
$H(x;\alpha,\beta) = -\ln(1 - F(x;\alpha,\beta))$

$$H(x;\alpha,\beta)=\ln\left(\left(\frac{\sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j+\beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}{\beta+1}\right)-1\right) \quad (3.4)$$

3.5 Odds rate function

The Odds rate function of the MLG distribution is defined as

$$O(x;\alpha,\beta)=\frac{F(x;\alpha,\beta)}{1-F(x;\alpha,\beta)}$$

$$O(x;\alpha,\beta)=\left(\frac{(\beta+1)-\sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j+\beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}{\left(\sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j+\beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}\right)}\right) \quad (3.5)$$

3.6 Mean Residual Life function

The mean residual life function of the MLG distribution is defined as

$$M(x;\alpha,\beta)=\frac{1}{S(x)}\int_x^{\infty}t\,f(t;\alpha,\beta)dt-x$$

$$M(x;\alpha,\beta)=\left(\frac{1}{1-\frac{1}{\beta+1}\left[1-\left(\sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j+\beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}\right)+\beta\right]} \times\int_x^{\infty}t\,\frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{t}{\beta}\right)^{-(\alpha+1)}+\frac{\alpha^2\beta}{2}t^2\,e^{-\alpha t}\right)dt\right)-x \quad (3.6)$$

First term integration

$$M(x)=\int_x^{\infty}t\,\frac{1}{\beta}\left(1+\frac{t}{\beta}\right)^{-(\alpha+1)}dt \quad (3.7)$$

Integration using the substitution method

By assuming, $u=1+\frac{t}{\beta}, then\ t=\beta(u-1)\ and\ dt=\beta du$

When $t=x, u=1+\frac{x}{\beta}\ and\ x\to\infty, u\to\infty$

Substitute the limits of integration and simplify the expressions in Eq. (3.7)

$$M(x;\alpha,\beta)=\beta\left(\frac{\left(1+\frac{x}{\beta}\right)^{-\alpha+1}}{(\alpha+1)}+\frac{\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\alpha}\right) \quad (3.8)$$

Second term integration

$$M(x;\alpha,\beta)=\frac{\alpha^2\beta}{2}\int_x^{\infty}t^3\,e^{-\alpha t}\,dt \quad (3.9)$$

Then, using the substitution method in Eq. (3.9)

By letting, $u=\alpha t, then\ t=\frac{u}{\alpha}\ and\ dt=\frac{du}{\alpha}$

When $t=x, u=\alpha x\ and\ x\to\infty, u\to\infty$

The upper incomplete gamma function is defined as

$$\Gamma(s,x) \ = \int_x^{\infty} t^{s-1} e^{-t} dt$$

To solve this integral, we recognize that it resembles the form of the upper incomplete gamma function in Eq. (3.9)

$$M(x;\alpha,\beta) = \frac{\beta\Gamma(4,\alpha x\ )}{2\alpha^2} \tag{3.10}$$

Combine these integrals in Eqs. (3.8) and (3.10), substituting in Eq. (3.6), and simplifying the expression given by

$$M(x;\alpha,\beta) = \frac{\alpha\beta}{\beta+1}\left(\frac{\left(1+\frac{x}{\beta}\right)^{-\alpha+1}}{(\alpha+1)} + \frac{\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\alpha} + \frac{\Gamma(4,\alpha t\ )}{2\alpha^2}\right) \tag{3.11}$$

So, the final result of the integral is

$$M(x;\alpha,\beta) = \left(\frac{\alpha\beta\left(\frac{\left(1+\frac{x}{\beta}\right)^{-\alpha+1}}{(\alpha+1)} + \frac{\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\alpha} + \frac{\Gamma(4,\alpha x\ )}{2\alpha^2}\right)}{\sum_{j=0}^{\infty}(-1)^j \binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j + \beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}\right) - x \tag{3.12}$$

4. Probabilistic Properties

In this section, we derived the statistical properties, moments, the moment-generating function, characteristic function, and r^{th} moment for the new MLG distribution of the random variable. Including the mean, variance, coefficient of variation, standard deviation, skewness, kurtosis, and dispersion investigated.

4.1 Moments

The r^{th} moments of a RV X of the MLG distribution are defined as

$$E(X^r) = \mu_r' = \int_0^{\infty} x^r \ f(x;\alpha,\beta)\, dx$$

$$E(X^r) = \int_0^{\infty} x^r \ \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\ e^{-\alpha x}\right) dx \tag{4.1}$$

$$E(X^r) = \int_0^{\infty} \frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} dx \tag{4.2}$$

Then, integration using the substitution meth

Let assuming, $u = 1+\frac{x}{\beta}, then\ x = \beta(u-1)\ and\ dx = \beta du$

When $x = 0, u = 1\ and\ x \to \infty, u \to \infty$

Let’s make a substitution to simplify the integral in Eq. (4.2)

$$E(X^r) = \int_1^{\infty} \left(\beta(u-1)\right)^r\ u^{-(\alpha+1)}\beta du$$

$$E(X^r) = \beta^{r+1}\int_1^{\infty} \frac{(u-1)^r}{u^{(\alpha+1)}} du \tag{4.3}$$

Then, using the following binomial expansion in Eq. (4.3)

$$(1-z)^{a}=\sum_{j=0}^{\infty}(-1)^{j}\binom{a}{j}\ z^{j} \tag{4.4}$$

Substituting this into the integral, Eq. (4.3), we get

$$E(X^{r})=\beta^{r+1}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\int_{1}^{\infty}u^{k}\,\frac{1}{u^{(\alpha+1)}}du$$

$$E(X^{r})=\beta^{r}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\left(-\frac{1}{k-\alpha}\right) \tag{4.5}$$

$$E(X^{r})=\int_{0}^{\infty}\frac{\alpha^{2}\beta}{2}x^{r+2}\,e^{-\alpha x}dx \tag{4.6}$$

Then, using the following gamma function is defined as in Eq. (4.6)

$$\int_{0}^{\infty}x^{z-1}e^{-px}\,dx=\frac{\Gamma(z)}{p^{z}}$$

$$E(X^{r})=\frac{\beta}{2}\frac{\Gamma(r+3)}{\alpha^{r+1}} \tag{4.7}$$

Combine these with Eqs. (4.5) and (4.7), and the simplified expression becomes

$$E(X^{r})=\frac{1}{\beta+1}\left(\alpha\beta^{r}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{\beta\Gamma(r+3)}{2\alpha^{r}}\right) \tag{4.8}$$

Where $\Gamma(.)$ Is the gamma function. Subsequently, the first moment (mean), second moment, third moment, and fourth moment can be defined by substituting $r=1,2,3,4$ in Eq. (4.8)

$$E(X)=\frac{\beta}{\beta+1}\left(\alpha\sum_{k=0}^{1}(-1)^{1-k}\binom{1}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{3}{\alpha}\right)$$

Simplifying this, we get

$$E(X)=mean=\frac{\beta}{\beta+1}\left(\frac{3-2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right) \tag{4.9}$$

$$E(X^{2})=\frac{\beta}{\beta+1}\left(\alpha\beta\sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{12}{\alpha^{2}}\right)$$

$$E(X^{3})=\frac{\beta}{\beta+1}\left(\alpha\beta^{2}\sum_{k=0}^{3}(-1)^{3-k}\binom{3}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{60}{\alpha^{3}}\right)$$

$$E(X^{4})=\frac{\beta}{\beta+1}\left(\alpha\beta^{3}\sum_{k=0}^{4}(-1)^{4-k}\binom{4}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{360}{\alpha^{4}}\right)$$

$$Variance=\sigma^{2}=E(X^{2})-\left(E(X)\right)^{2}$$

$$\sigma^{2}=\left(\frac{\beta}{\beta+1}\left(\alpha\beta\sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{12}{\alpha^{2}}\right)\right)-\left(\frac{\beta}{\beta+1}\left(\alpha\sum_{k=0}^{1}(-1)^{1-k}\binom{1}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{3}{\alpha}\right)\right)^{2} \tag{4.10}$$

First term, Simplification in Eq. (4.10)

$$=\frac{\beta}{\beta+1}\left(\alpha\beta\sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{12}{\alpha^{2}}\right)$$

$$=\left(\frac{\beta}{\beta+1}\left(\alpha\beta\left(\binom{2}{0}-\frac{1}{0-\alpha}+\binom{2}{1}-\frac{1}{1-\alpha}+\binom{2}{2}-\frac{1}{2-\alpha}\right)+\frac{12}{\alpha^2}\right)\right)$$

$$=\left(\frac{\beta^2}{\beta+1}\left(1-\frac{2\alpha}{1-\alpha}+\frac{\alpha}{\alpha-2}\right)+\frac{12\beta}{\alpha^2(\beta+1)}\right)$$

Simplifying this, we get

$$=\left(\frac{\beta^2(\beta+1)}{(\beta+1)^2}\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}\right)+\frac{12\beta^2(\beta+1)}{\alpha^2\beta(\beta+1)^2}\right)$$

$$=\frac{\beta^2}{(\beta+1)^2}\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}+\frac{12}{\alpha^2\beta}\right) \tag{4.11}$$

Second terms, Simplification in Eq. (4.10)

$$=\left(\frac{\beta}{\beta+1}\left(\alpha\sum_{k=0}^{1}(-1)^{1-k}\binom{1}{k}\left(-\frac{1}{k-\alpha}\right)+\frac{3}{\alpha}\right)\right)^2$$

$$=\left(\frac{\beta}{\beta+1}\left(\alpha\left(\binom{1}{0}-\frac{1}{0-\alpha}+\binom{1}{1}-\frac{1}{1-\alpha}\right)+\frac{3}{\alpha}\right)\right)^2$$

$$=\left(\frac{\beta}{\beta+1}\left(\alpha\left(\frac{-1}{-\alpha}-\frac{1}{1-\alpha}\right)+\frac{3}{\alpha}\right)\right)^2$$

$$=\left(\frac{\beta}{\beta+1}\left(\frac{3-2\alpha^2-2\alpha}{\alpha(1-\alpha)}\right)\right)^2$$

$$=\frac{\beta^2}{(\beta+1)^2}\left(\frac{(3-2\alpha^2-2\alpha)^2}{\alpha^2(1-\alpha)^2}\right)$$

Simplifying this, we get

$$=\frac{\beta^2}{(\beta+1)^2}\left(\frac{8\alpha^3+20\alpha^2-24\alpha+9}{\alpha^4+\alpha^2-2\alpha^3}\right) \tag{4.12}$$

Therefore, combining Eqs. (4.11) and (4.12) and these results, we have

$$\sigma^2=\frac{\beta^2}{(\beta+1)^2}\left(\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}+\frac{12}{\alpha^2\beta}\right)-\left(\frac{8\alpha^3+20\alpha^2-24\alpha+9}{\alpha^4+\alpha^2-2\alpha^3}\right)\right) \tag{4.13}$$

$$SD\sigma=\frac{\beta}{\beta+1}\sqrt{\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}+\frac{12}{\alpha^2\beta}\right)-\left(\frac{8\alpha^3+20\alpha^2-24\alpha+9}{\alpha^4+\alpha^2-2\alpha^3}\right)}$$

4.2 Coefficient of Variation

The coefficient of variation of the MLG distribution is defined as

$$C.V\left(\frac{\sigma}{\mu}\right)=\left(\frac{\sqrt{\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}+\frac{12}{\alpha^2\beta}\right)-\left(\frac{8\alpha^3+20\alpha^2-24\alpha+9}{\alpha^4+\alpha^2-2\alpha^3}\right)}}{\left(\frac{3-2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right)}\right) \tag{4.14}$$

4.3 Skewness

The skewness of the MLG distribution is defined as

$$Sk(X)=\sqrt{\beta_1}=\frac{E(X^3)}{\left(var(X)\right)^{\frac{3}{2}}}$$

After simplification, we get

$$Sk(X)=\frac{(\beta+1)^2}{\beta^2}\left(\frac{\left(\alpha\beta^2\left(-1+\frac{3\alpha}{1-\alpha}-\frac{3\alpha}{2-\alpha}+\frac{\alpha}{3-\alpha}\right)+\frac{60}{\alpha^3}\right)}{\left(\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}+\frac{12}{\alpha^2\beta}\right)-\left(\frac{8\alpha^3+20\alpha^2-24\alpha+9}{\alpha^4+\alpha^2-2\alpha^3}\right)\right)^{\frac{3}{2}}}\right) \tag{4.15}$$

4.4 Kurtosis

The kurtosis of the MLG distribution is defined as

$$Ku(X)=\beta_1=\frac{E(X^4)}{\left(var(X)\right)^2}$$

$$Ku(X)=\frac{(\beta+1)^3}{\beta^3}\left(\frac{\left(\alpha\beta^3\left(-1+\frac{4\alpha}{1-\alpha}-\frac{6\alpha}{2-\alpha}+\frac{4\alpha}{3-\alpha}-\frac{\alpha}{4-\alpha}\right)+\frac{360}{\alpha^4}\right)}{\left(\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}+\frac{12}{\alpha^2\beta}\right)-\left(\frac{8\alpha^3+20\alpha^2-24\alpha+9}{\alpha^4+\alpha^2-2\alpha^3}\right)\right)^2}\right) \tag{4.16}$$

4.5 Dispersion

The dispersion of the MLG distribution is defined as

Dispersion= $\frac{\sigma^2}{\mu}$

$$=\frac{\beta}{\beta+1}\left(\frac{\left(\left(\frac{-2(2\alpha^2+1-4\alpha)}{(1-\alpha)(\alpha-2)}+\frac{12}{\alpha^2\beta}\right)-\left(\frac{8\alpha^3+20\alpha^2-24\alpha+9}{\alpha^4+\alpha^2-2\alpha^3}\right)\right)}{\left(\frac{3-2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right)}\right) \tag{4.17}$$

4.6 Moment Generating Function

The moment generating function (MGF) of a RV X is denoted by $M_X(t)$ of the MLG distribution is defined as

$$M_X(t)=E(e^{tX})=\int_0^\infty e^{tx}\,f(x;\alpha,\beta)\;\;dx\;\;,t\epsilon\mathcal{R}$$

$$M_X(t)=\int_0^\infty e^{tx}\,\frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}+\frac{\alpha^2\beta}{2}x^2\,e^{-\alpha x}\right)dx \tag{4.18}$$

First term integration,

$$M_X(t)=\int_0^\infty\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}dx \tag{4.19}$$

Using the substitution in Eq. (4.19)

Let assuming, $u=1+\frac{x}{\beta}$, which implies $x=\beta(u-1)$and $dx=\beta du$.

Then, $x=0, u=1\;and\;x\to\infty, u\to\infty$

Let's make a substitution to simply the integral in Eq. (4.19)

$$M_X(t)=\int_1^\infty u^{-(\alpha+1)}\,e^{t\beta(u-1)}\,\beta du$$

$$M_X(t) = \beta e^{-t\beta} \int_1^{\infty} u^{-(\alpha+1)} e^{t\beta u} \, du \qquad (4.20)$$

The Laplace transform of a function $f(t)$is given by in Eq. (4.20)

$$L(f(t)) = F(s) = \int_1^{\infty} e^{-st} f(t) \, dt \qquad (4.21)$$

the Laplace transform of $e^{t\beta u-(\alpha+1)\ln(u)}$ with respect to u.

$F(s) = L(e^{t\beta u-(\alpha+1)\ln(u)})$

find $F(s)$ using the definition of the Laplace transform

$$F(s) = \int_0^{\infty} e^{-su} e^{t\beta u-(\alpha+1)\ln(u)} \, du$$

$$M_X(t) = \frac{1}{\beta}\beta e^{-t\beta} \frac{\Gamma(\alpha+2)}{(s-t\beta)^{\alpha+2}}$$

$$M_X(t) = \frac{\Gamma(\alpha+2)}{(s-t\beta)^{\alpha+2}} e^{-t\beta} \qquad (4.22)$$

Second term integration in Eq. (4.18)

$$M_X(t) = \int_0^{\infty} e^{tx} \frac{\alpha^2\beta}{2} x^2 e^{-\alpha x} \, dx \qquad (4.22)$$

Substituting this back into the integral in Eq. (4.22)

By letting, $u = x^2 \quad and \quad dv = -\frac{e^{-(\alpha-t)x}}{(\alpha-t)}$

Let's make a substitution to simplify the integral

$$M_X(t) = \frac{\alpha^2\beta}{(\alpha-t)^3} \qquad (4.23)$$

Now, let's add the integral in Eqs. (4.22) and (4.23) together.

$$M_X(t) = \frac{\alpha}{\beta+1}\left(\frac{\alpha^2\beta}{(\alpha-t)^3} + \frac{\Gamma(\alpha+2)}{(s-t\beta)^{\alpha+2}} e^{-t\beta}\right) \qquad (4.24)$$

4.7 Characteristics Function

The characteristics function (CF) of a RV X, it is denoted by $\phi_X(t)$ and MLG distribution is defined as

$$\phi_X(t) = E(e^{itX}) = \int_0^{\infty} e^{itx} f(x;\alpha,\beta) \, dx$$

$$\phi_X(t) = M_X(it)$$

$$\phi_X(t) = \frac{\alpha}{\beta+1}\left(\frac{\alpha^2\beta}{(\alpha-it)^3} + \frac{\Gamma(\alpha+2)}{(s-it\beta)^{\alpha+2}} e^{-it\beta}\right) \qquad (4.25)$$

5. Hormonic Mean

If H_X is the harmonic mean (HM) of the RV X, and MLG distribution is defined as

$$H_X = E\left(\frac{1}{X}\right)$$

$$H.M = \int_0^{\infty} \frac{1}{x} \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2} x^2 e^{-\alpha x}\right) dx \qquad (5.1)$$

First term integration,

$$H.M = \int_0^{\infty} x^{-1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\right) dx \qquad (5.2)$$

Using the substitution, $u = 1 + \frac{x}{\beta}$, which implies $x = \beta(u-1)$and $dx = \beta du$.

When, $x = 0, u = 1$ and $x \to \infty, u \to \infty$
Substituting the limits of integration and simplify the expression in Eq. (5.2)

$$H.M = \frac{1}{\beta}\int_{1}^{\infty}(u-1)^{-1}\ u^{-(\alpha+1)}du \tag{5.3}$$

Then, using the following geometric series is defined as

$$(1-x)^{-1} = \sum_{j=0}^{\infty} x^{j} \tag{5.4}$$

So, the final expressions for the integral are

$$H.M = \frac{1}{\beta}\sum_{j=0}^{\infty}\frac{1}{(\alpha-j)} \tag{5.5}$$

Second term integration

$$H.M = \frac{\alpha^2\beta}{2}\int_{0}^{\infty} x^2\ e^{-\alpha x}\ dx \tag{5.6}$$

Using the integration by substitution method in Eq. (5.6)
$u = x$ and $u = x^{\alpha}\quad dv = e^{-\theta x}$
Simplifying this, we get

$$H.M = \frac{\beta}{2} \tag{5.7}$$

So, the final results of the integral are

$$H.M = \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\sum_{j=0}^{\infty}\frac{1}{(\alpha-j)}+\frac{\beta}{2}\right) \tag{5.8}$$

6. Mean Deviation

The Mean deviation (MD) of the RV X, and MLG distribution is defined as

$$D(\mu) = E(|X-\mu|)$$

$$D(\mu) = \int_{0}^{\infty}|X-\mu|\ \ f(x;\alpha,\beta)dx$$

$$D(\mu) = \int_{0}^{\mu}(\mu-x)\ f(x;\alpha,\beta)\ dx + \int_{\mu}^{\infty}(x-\mu)\ \ f(x;\alpha,\beta)\ dx$$

Simplifying this, we get

$$D(\mu) = 2\mu F(\mu) - 2\int_{0}^{\mu} x\ \ f(x;\alpha,\beta)\ dx \tag{6.1}$$

Then,

$$D(\mu) = \int_{0}^{\mu} x\ \ f(x;\alpha,\beta)\ dx$$

$$D(\mu) = \frac{\alpha}{\beta+1}\int_{0}^{\mu} x\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}+\frac{\alpha^2\beta}{2}x^2\ e^{-\alpha x}\right)dx \tag{6.2}$$

First term integration

$$D(\mu) = \frac{1}{\alpha}\int_{0}^{\mu} x\ \left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\ dx \tag{6.3}$$

Then, using the substitution in Eq. (6.3), $u = 1+\frac{x}{\beta}$, which implies $x = \beta(u-1)$and $dx = \beta du$.

$Then,\qquad x = 0,\qquad u = 1$ and $x = \mu,\qquad u = 1+\frac{\mu}{\beta}$

Let's make a substitution to simplify the integral in Eq. (6.3)

$$D(\mu) = \beta\left(\left(\frac{\left(1+\frac{\mu}{\beta}\right)^{-\alpha}-1}{\alpha}\right)-\left(\frac{\left(1+\frac{\mu}{\beta}\right)^{-\alpha+1}-1}{\alpha+1}\right)\right) \quad (6.4)$$

For the second term integration to simplify the expression

$$D(\mu) = \frac{\alpha^2\beta}{2}\int_0^{\mu} x^3\, e^{-\alpha x}\, dx \quad (6.5)$$

Then, using the substitution method in Eq. (6.5)

By letting, $u = x^3\,,\ \ dv = e^{-\alpha x}\,dx,\ \ then\ \ du = 3x^2\,dx,\ \ v = -\frac{1}{\alpha}e^{-\alpha x}$

Substituting this back into the integral, we have

$$D(\mu) = \beta\left(-\left(\frac{\mu^2}{2}(\alpha\mu+3)+\frac{3}{\alpha}\left(\mu+\frac{1}{\alpha}\right)\right)e^{-\alpha\mu}+\frac{3}{\alpha^2}\right) \quad (6.6)$$

Combine result these, the simplified expression becomes

$$D(\mu) = \frac{2\beta}{\beta+1}\left\{\begin{matrix}\left(\frac{3-2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right)1-\left(\frac{\sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j-1}{j}\left(\frac{\mu}{\beta}\right)^j+\beta\left(\left(\frac{\alpha\mu}{2}+1\right)\alpha\mu+1\right)e^{-\alpha\mu}}{\beta+1}\right)\\ -\alpha\left(\left(\frac{\left(1+\frac{\mu}{\beta}\right)^{-\alpha}-1}{\alpha}\right)+\frac{3}{\alpha^2}-\left(\frac{\left(1+\frac{\mu}{\beta}\right)^{-\alpha+1}-1}{\alpha+1}\right)-\left(\frac{\mu^2}{2}(\alpha\mu+3)+\frac{3}{\alpha}\left(\mu+\frac{1}{\alpha}\right)\right)e^{-\alpha\mu}\right)\end{matrix}\right\} \quad (6.7)$$

7. Median

The mean deviation from median of the RV X, and MLG distribution is defined as

$$D(M) = E(|X-M|)$$

$$D(M) = \int_0^{\infty}|X-M|\ f(x;\alpha,\beta)dx$$

$$D(M) = \int_0^{M}(M-x)f(x;\alpha,\beta)\,dx + \int_M^{\infty}(x-M)f(x;\alpha,\beta)\,dx$$

Simplifying this, we get

$$D(M) = \mu - 2\int_0^{M} x\, f(x;\alpha,\beta)\,dx \quad (7.1)$$

Then,

$$D(M) = \int_0^{M} x\ f(x;\alpha,\beta)\,dx$$

$$D(M) = \frac{\alpha}{\beta+1}\int_0^{M} x\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}+\frac{\alpha^2\beta}{2}x^2\,e^{-\alpha x}\right)dx \quad (7.2)$$

First term integration

$$D(M) = \frac{1}{\alpha}\int_0^{M} x\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}dx \quad (7.3)$$

Integration, using the substitution in Eq. (7.3),

Let assuming, $u = 1+\frac{x}{\beta}$, which implies $x = \beta(u-1)$ and $dx = \beta du$.

$$when, \quad x = 0, \quad u = 1 \; and \; x = M, \quad u = 1 + \frac{M}{\beta}.$$

Let's make a substitution to simplify the integral.

$$D(M) = \beta\left(\left(\frac{\left(1+\frac{M}{\beta}\right)^{-\alpha} - 1}{\alpha}\right) - \left(\frac{\left(1+\frac{M}{\beta}\right)^{-\alpha+1} - 1}{\alpha+1}\right)\right) \tag{7.4}$$

For the second term integration to simplify the expression

$$D(M) = \frac{\alpha^2\beta}{2}\int_0^M x^3\, e^{-\alpha x}\, dx \tag{7.5}$$

By letting, $u = x^3$, $dv = e^{-\alpha x}\, dx$, $then \;\; du = 3x^2\, dx$, $v = -\frac{1}{\alpha} e^{-\alpha x}$

Substituting in Eq. (7.5), this back into the integral, we have

$$D(M) = \beta\left(-\left(\frac{M^2}{2}(\alpha M + 3) + \frac{3}{\alpha}\left(M + \frac{1}{\alpha}\right)\right)e^{-\alpha M} + \frac{3}{\alpha^2}\right) \tag{7.6}$$

Therefore, combining the Eqs. (7.4) and (7.6) results, we have

$$D(M) = \frac{\beta}{\beta+1}\left(\left(\frac{3 - 2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right) - 2\alpha\left(\left(\frac{\left(1+\frac{M}{\beta}\right)^{-\alpha} - 1}{\alpha}\right) + \frac{3}{\alpha^2} - \left(\frac{\left(1+\frac{M}{\beta}\right)^{-\alpha+1} - 1}{\alpha+1}\right) - \left(\frac{M^2}{2}(\alpha M + 3) + \frac{3}{\alpha}\left(M + \frac{1}{\alpha}\right)\right)e^{-\alpha M}\right)\right) \tag{7.8}$$

8. Order Statistics

The probability density function pdf of the r^{th} order statistics of the new mixture of Lomax and gamma distribution is derived. Let $X_1, X_2, \dots, X_n$ be a simple random sample from MLG distribution with cdf and pdf given by (9) and (1), respectively. Let $X_{(1:n)} \le X_{(2:n)} \le \cdots \le X_{(n:n)}$ denote the order statistics defined from this sample. We now given the pdf of $X_{r:n}$, say $f_{r;n}(x)$ of $X_{r:n}$, $r = 1,2,\dots,n$. The probability density function pdf of the new mixture of Lomax and gamma distribution the r^{th} order statistics $X_{r:n}$, $r = 1,2,\dots,n$ is defined as

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}\left(F(x;\alpha,\beta)\right)^{r-1}\left(1 - F(x;\alpha,\beta)\right)^{n-r} f(x;\alpha,\beta), x > 0, \alpha,\beta > 0 \tag{8.1}$$

Where $F(.)$ and $f(.)$ are given by (2.10) and (2.1) respectively,

$$and \;\; Z_{\,r:n} = \frac{n!}{(r-1)!\,(n-r)!}$$

$$f_{r:n} = Z_{\,r:n}\left(F(x;\alpha,\beta)\right)^{r-1}\left(1 - F(x;\alpha,\beta)\right)^{n-r} f(x;\alpha,\beta) \tag{8.2}$$

Then, using the following binomial series expansion in Eq. (8.2)

$$(1-z)^a = \sum_{j=0}^{\infty}(-1)^j \binom{a}{j} z^j \tag{8.3}$$

$$f_{r:n} = Z_{r:n} \sum_{s=0}^{\infty} (-1)^s \binom{n-r}{s} \left(F(x;\alpha,\beta)\right)^{r+s-1} f(x;\alpha,\beta)$$

$$f_{r:n} = \left(\begin{array}{c} Z_{r:n} \sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} (-1)^{s+i+k} \binom{n-r}{s}\binom{r+s-1}{i}\binom{i}{j}\binom{k+\alpha j-1}{k} \\ \binom{i-j}{l}\binom{l}{m}\left(\frac{1}{\beta+1}\right)^i \left(\frac{1}{\beta}\right)^k \left(\frac{1}{2}\right)^m \frac{\left((l+m)\ln\alpha\right)^n}{n!} \frac{\left((i-j)\ln\beta\right)^p}{p!} x^{k+l+m} e^{-\alpha i x} \\ \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2} x^2 e^{-\alpha x}\right) \end{array} \right)$$

First order statistics

$$f_{1:n} = \left(\begin{array}{c} Z_{r:n} \sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} (-1)^{s+i+k} \binom{n-1}{s}\binom{s}{i}\binom{i}{j}\binom{k+\alpha j-1}{k} \\ \binom{i-j}{l}\binom{l}{m}\left(\frac{1}{\beta+1}\right)^i \left(\frac{1}{\beta}\right)^k \left(\frac{1}{2}\right)^m \frac{\left((l+m)\ln\alpha\right)^n}{n!} \frac{\left((i-j)\ln\beta\right)^p}{p!} x^{k+l+m} e^{-\alpha i x} \\ \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2} x^2 e^{-\alpha x}\right) \end{array} \right) \quad (8.4)$$

n^{th} order statistics

$$f_{n:n} = \left(\begin{array}{c} Z_{r:n} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} (-1)^{s+i+k} \binom{n+s-1}{i}\binom{i}{j}\binom{k+\alpha j-1}{k} \\ \binom{i-j}{l}\binom{l}{m}\left(\frac{1}{\beta+1}\right)^i \left(\frac{1}{\beta}\right)^k \left(\frac{1}{2}\right)^m \frac{\left((l+m)\ln\alpha\right)^n}{n!} \frac{\left((i-j)\ln\beta\right)^p}{p!} x^{k+l+m} e^{-\alpha i x} \\ \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2} x^2 e^{-\alpha x}\right) \end{array} \right) \quad (8.5)$$

8.1 Quantile function

The quantile function of a distribution with cdf, $F(x;\alpha,\beta)$, is defined by $q = F\left(x_q;\alpha,\beta\right)$, where $0 < q < 1$. Thus, the quantile function of MLG distribution is given by

$$1-q = \left(\frac{\sum_{j=0}^{\infty}(-1)^j \binom{\alpha+j-1}{j}\left(\frac{x}{\beta}\right)^j + \beta\left(\left(\frac{\alpha x}{2}+1\right)\alpha x+1\right)e^{-\alpha x}}{\beta+1} \right) \quad (8.6)$$

9. Entropies

In this section, we derived the Rényi entropy, and Tsallis entropy from the new MLG distribution. It is well known that entropy and information can be considered measures of uncertainty or the randomness of a probability distribution. It is applied in many fields, such as engineering, finance, information theory, and biomedicine. The entropy functionals for probability distribution were derived on the basis of a variational definition of uncertainty measure.

9.1 Rényi Entropy

Entropy is defined as a random variable X is a measure of the variation of the uncertainty. It is used in many fields, such as engineering, statistical mechanics, finance, information theory, biomedicine, and economics. The entropy measure is the Rényi of order which is defined as

$$R_\gamma = \frac{1}{1-\gamma}\log\left(\int_0^\infty [f(x;\alpha,\beta)]^\gamma\,dx\right) \qquad ;\gamma>0,\gamma\neq 1$$

$$R_\gamma = \frac{1}{1-\gamma}\log\left(\int_0^\infty \left(\frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\,e^{-\alpha x}\right)\right)^\gamma dx\right) \tag{9.1}$$

Using the following binomial series expansion in Eq. (9.1), we get

$$(a+b)^z = \sum_{j=0}^{\infty}\binom{z}{j}\,(a)^j b^{z-j} \tag{9.2}$$

Then, binomial series expansion and, simplify the expression in Eq. (9.1)

$$(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k \tag{9.3}$$

$$R_\gamma = \frac{1}{1-\gamma}\log\left(\begin{array}{c}\left(\frac{\alpha}{\beta+1}\right)^\gamma \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\gamma}{j}(-1)^k\binom{(\alpha+1)j+k-1}{k}\left(\frac{1}{\beta}\right)^{j+k}\left(\frac{\alpha^2\beta}{2}\right)^{\gamma-j} \\ \times\int_0^\infty x^{2(\gamma-j)+k}e^{-\alpha(\gamma-j)x}\end{array}\right) \tag{9.4}$$

Then Eq. (9.4), simplify the integral using the gamma function is given

$$R_\gamma = \frac{1}{1-\gamma}\log\left(\begin{array}{c}\left(\frac{\alpha}{\beta+1}\right)^\gamma \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\gamma}{j}(-1)^k\binom{(\alpha+1)j+k-1}{k}\left(\frac{1}{\beta}\right)^{j+k}\left(\frac{\alpha^2\beta}{2}\right)^{\gamma-j} \\ \times\frac{\Gamma(2(\gamma-j)+k+1)}{\left(\alpha(\gamma-j)\right)^{2(\gamma-j)+k+1}}\end{array}\right) \tag{9.5}$$

9.2 Tsallis Entropy

The Boltzmann-Gibbs (B-G) statistical properties initiated by Tsallis have received a great deal of attention. This generalization of (B-G) statistics was first proposed by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for continuous random variables, which is defined as

$$T_\gamma = \frac{1}{\gamma-1}\left(1-\int_0^\infty [\,f(x;\alpha,\beta)]^\gamma\;dx\right) \qquad ;\gamma>0,\gamma\neq 1$$

$$T_\gamma = \frac{1}{\gamma-1}\left(1-\int_0^\infty \left(\frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\,e^{-\alpha x}\right)\right)^\gamma dx\right) \tag{9.6}$$

Solving the integration and the simplified expression becomes in Eq. (9.6)

$$T_\gamma = \frac{1}{\gamma-1}\left(1-\left(\begin{array}{c}\left(\frac{\alpha}{\beta+1}\right)^\gamma \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\gamma}{j}(-1)^k\binom{(\alpha+1)j+k-1}{k}\left(\frac{1}{\beta}\right)^{j+k}\left(\frac{\alpha^2\beta}{2}\right)^{\gamma-j} \\ \times\frac{\Gamma(2(\gamma-j)+k+1)}{\left(\alpha(\gamma-j)\right)^{2(\gamma-j)+k+1}}\end{array}\right)\right) \tag{9.7}$$

10. Stochastic Ordering

A crucial technique in reliability and finance for evaluating the relative performance of the models is stochastic ordering. Let X and Y be two random variables of the new mixture of Lomax and gamma distribution with pdf, cdf, and reliability functions $f(x), f(y), F(x), F(y). S(x) = 1-F(x), and\ F(y).$

1- Likelihood ratio order ($X \leq_{LR} Y$) if $\frac{f_X(x;\alpha,\beta)}{f_Y(x;\lambda,\delta)}$ decreases in x
2- Stochastic order ($X \leq_{ST} Y$) if $F_X(x;\alpha,\beta) \geq F_Y(x;\lambda,\delta)\ \forall\ x$

3- Hazard rate order $(X \leq_{HR} Y)$ if $h_X(x;\alpha,\beta) \geq h_Y(x;\lambda,\delta)\ \forall\ x$
4- Mean residual life order $(X \leq_{MRL} Y)$ if $MRL_X(x;\alpha,\beta) \leq MRL_Y(x;\lambda,\delta)\ \forall\ x$

Prove that the mixture of Lomax and gamma distribution complies with the ordering with the highest likelihood (the likelihood ratio ordering).

Assume that X and Y are two independent Random variables with probability distribution function of the new mixture of Lomax and gamma distribution $f_X(x;\alpha,\beta)$and $f_Y(x;\lambda,\delta)$If $\alpha < \lambda$ and $\delta < \beta$, then

$$\Lambda = \frac{f_X(x;\alpha,\beta)}{f_Y(x;\lambda,\delta)}$$

$$\Lambda = \left(\frac{\frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\,e^{-\alpha x}\right)}{\frac{\lambda}{\delta+1}\left(\frac{1}{\delta}\left(1+\frac{x}{\delta}\right)^{-(\lambda+1)} + \frac{\lambda^2\delta}{2}x^2\,e^{-\lambda x}\right)} \right) \tag{10.1}$$

Therefore, the log-likelihood function is given in Eq. (10.1)

$$\log[\Lambda] = \log\left[\frac{\alpha(\delta+1)}{\lambda(\zeta+1)}\right] + \log\left[\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\,e^{-\alpha x}\right] - \log\left[\frac{1}{\delta}\left(1+\frac{x}{\delta}\right)^{-(\lambda+1)} + \frac{\lambda^2\delta}{2}x^2\,e^{-\lambda x}\right] \tag{10.2}$$

Differentiating with respect to x, in Eq. (10.2)

$$\frac{\partial\log[\Lambda]}{\partial x} = \left(\frac{\left[-\frac{(\alpha+1)}{\alpha\beta}\left(\left(1+\frac{x}{\beta}\right)^{-(\alpha+2)}\right) + \alpha^2\beta e^{-\alpha x} - \frac{\alpha^3\beta}{2}x^2 e^{-\alpha x}\right]}{\left[\frac{1}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x^2\,e^{-\alpha x}\right]} - \frac{\left[-\frac{(\lambda+1)}{\lambda\delta}\left(\left(1+\frac{x}{\delta}\right)^{-(\lambda+2)}\right) + \lambda^2\delta e^{-\lambda x} - \frac{\lambda^3\delta}{2}x^2 e^{-\lambda x}\right]}{\left[\frac{1}{\delta}\left(1+\frac{x}{\delta}\right)^{-(\lambda+1)} + \frac{\lambda^2\delta}{2}x^2\,e^{-\lambda x}\right]} \right) \tag{10.3}$$

Hence, $\frac{\partial\log[\Lambda]}{\partial x} < 0,$ **if** $\boldsymbol{\alpha} < \boldsymbol{\lambda}, \boldsymbol{\beta} < \delta$.

$X \leq_{LR} Y \Rightarrow X \leq_{HR} Y \Rightarrow X \leq_{MRL}\ Y$ *and* $X \leq_{HR}\ Y \Rightarrow X \leq_{ST} Y$.

11. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have been obtained using the MLG distribution in this section. The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures for a X be the random variable of a unit and $f(x;\alpha,\beta)$ be the probability density function of x. $f(x;\alpha,\beta)dx$ will be represented by the probability that a unit selected at random is defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\,f(x;\alpha,\beta)dx \quad and$$

$$L(p) = \frac{1}{\mu}\int_0^q x\,f(x;\alpha,\beta)dx$$

Where, $q = F^{-1}(p);\quad q\epsilon[0,1]$ and $\mu = E(X)$

$$\mu = E(X) = \frac{\beta}{\beta+1}\left(\frac{3-2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right)$$

$$B(p)=\frac{1}{p\mu}\int_0^q x\ \frac{\alpha}{\beta+1}\left(\frac{1}{\beta}\ \left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}+\frac{\alpha^2\beta}{2}x^2\ e^{-\alpha x}\right)dx \quad (11.1)$$

First term integration

$$B(p)=\frac{1}{\beta}\int_0^q x\ \left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\ dx \quad (11.2)$$

Using the substitution in Eq. (11.2), $u=1+\frac{x}{\beta}$, which implies $x=\beta(u-1)$and $dx=\beta du$.

$$Then,\qquad x=0,\qquad u=1\ and,\qquad x=q,\qquad u=1+\frac{q}{\beta}$$

Substitute the limits of integration and simplify the expression.

$$B(p)=\frac{1}{\beta}\int_1^{1+\frac{q}{\beta}}\beta(u-1)\ \ u^{-(\alpha+1)}\ \beta du$$

$$B(p)=\beta\left(\left(\frac{\left(1+\frac{q}{\beta}\right)^{-\alpha+1}-1}{-\alpha+1}\right)+\left(\frac{\left(1+\frac{q}{\beta}\right)^{-\alpha}-1}{\alpha}\right)\right) \quad (11.3)$$

Second term integration

$$B(p)=\frac{\alpha^2\beta}{2}\int_0^q x^3\ e^{-\alpha x}\ \ dx \quad (11.4)$$

$$Let,\ \ u=x^3\ ,\ \ dv=e^{-\alpha x}\,dx,\ \ \ then\ \ \ du=3x^2\ dx,\ \ v=-\frac{1}{\alpha}e^{-\alpha x}$$

Substitution in Eq. (11.4), to simplify the integral

$$B(p)=\beta\left(-\left(\frac{q^2}{2}(\alpha q+3)+\frac{3}{\alpha}\left(q+\frac{1}{\alpha}\right)\right)e^{-\alpha q}+\frac{3}{\alpha^2}\right) \quad (11.5)$$

Let's add these integrals in Eqs, (11.3) and (11.5) substituting in Eq. (11.1), and simplify the expression is

$$B(p)=\left(\frac{\alpha\left(\left(\frac{\left(1+\frac{q}{\beta}\right)^{-\alpha+1}-1}{-\alpha+1}\right)+\left(\frac{\left(1+\frac{q}{\beta}\right)^{-\alpha}-1}{\alpha}\right)-\left(\frac{q^2}{2}(\alpha q+3)+\frac{3}{\alpha}\left(q+\frac{1}{\alpha}\right)\right)e^{-\alpha q}+\frac{3}{\alpha^2}\right)}{p\left(\frac{3-2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right)}\right) \quad (11.6)$$

$$L(p)=pB(p)$$

$$L(p)=\left(\frac{\alpha\left(\left(\frac{\left(1+\frac{q}{\beta}\right)^{-\alpha+1}-1}{-\alpha+1}\right)+\left(\frac{\left(1+\frac{q}{\beta}\right)^{-\alpha}-1}{\alpha}\right)-\left(\frac{q^2}{2}(\alpha q+3)+\frac{3}{\alpha}\left(q+\frac{1}{\alpha}\right)\right)e^{-\alpha q}+\frac{3}{\alpha^2}\right)}{\left(\frac{3-2\alpha(\alpha+1)}{\alpha(1-\alpha)}\right)}\right) \quad (11.7)$$

12. Estimation of Parameters

The MLG distribution parameter's maximum likelihood estimates and Fisher's information matrix are provided in this section.

12.1 Maximum Likelihood Estimation (MLE) and Fisher's Information Matrix

Consider $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the new mixture of Lomax and gamma distribution with parameter α, β the log-likelihood function, which is defined as

$$L = (x; \alpha, \beta) = \prod_{i=1}^{n} f(x_i; \alpha, \beta)$$

$$L = \prod_{i=1}^{n} \frac{\alpha}{\beta + 1}\left(\frac{1}{\beta}\left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2 \beta}{2} x_i^2 \, e^{-\alpha x_i}\right) \tag{12.1}$$

Then, the log-likelihood function is given in Eq. (12.1)

$$\ell = \log L = n\log(\alpha) - n\log(\beta + 1) + \log \sum_{i=1}^{n}\left(\frac{1}{\beta}\left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2 \beta}{2} x_i^2 \, e^{-\alpha x_i}\right) \tag{12.2}$$

Differentiating with respect to $\alpha \; and \; \beta$ in Eq. (12.2)

$$\frac{\partial \log L}{\partial \alpha} = n\left(\frac{1}{\alpha}\right) + \sum_{i=1}^{n}\left(\frac{\left(\frac{\ln\left(1 + \frac{x_i}{\beta}\right)}{\beta} . \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} + \alpha\beta x_i^2 \, e^{-\alpha x_i} - \frac{\alpha^2 \beta}{2} x_i^3 \, e^{-\alpha x_i}\right)}{\left(\frac{1}{\beta}\left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2 \beta}{2} x_i^2 \, e^{-\alpha x_i}\right)}\right) = 0 \tag{12.3}$$

$$\frac{\partial \log L}{\partial \beta} = -n\left(\frac{1}{\beta + 1}\right) + \sum_{i=1}^{n}\left(\frac{\left(\frac{x_i}{\beta^4}\left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2}{2} x_i^2 e^{-\alpha x_i}\right)}{\left(\frac{1}{\beta}\left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2 \beta}{2} x_i^2 \, e^{-\alpha x_i}\right)}\right) = 0 \tag{12.4}$$

The maximum likelihood estimate of the parameters for the MLG distribution is provided by equations (12.3) and (12.4). The equation, however, cannot be solved analytically, so we used R programming and a data set to solve it numerically.

The asymptotic normality results are used to derive the confidence interval. Given that if $\hat{\lambda} = (\hat{\alpha}, , \hat{\beta})$ represents the MLE of $\lambda = (\alpha, \beta)$, the results can be expressed as follows:

$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_2(0, I^{-1}(\lambda))$

In this case, $I(\lambda)$represents Fisher's Information Matrix.

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) \end{pmatrix}$$

$$\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = -n\left(\frac{1}{\alpha^2}\right) + \sum_{i=1}^{n}\left(\frac{\left(\frac{1}{\beta}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x_i^2\, e^{-\alpha x_i}\right)\left(\beta x_i^3\, e^{-\alpha x_i}\left(\alpha\left(1-\frac{\alpha x_i}{2}\right)-(1+\alpha)\right)\right) - \left(\frac{\ln\left(1+\frac{x_i}{\beta}\right)}{\beta}.\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} + \alpha\beta x_i^2\, e^{-\alpha x_i} - \frac{\alpha^2\beta}{2}x_i^3\, e^{-\alpha x_i}\right)^2}{\left(\frac{1}{\beta}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x_i^2\, e^{-\alpha x_i}\right)^2}\right) \quad (12.5)$$

$$\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) = n\left(\frac{1}{(\beta+1)^2}\right) + \sum_{i=1}^{n}\left(\frac{\left(\frac{1}{\beta}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x_i^2\, e^{-\alpha x_i}\right)\left(\frac{1}{\beta^5}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)}\left(-4x_i+\frac{x_i^2}{\beta}\right)\right) - \left(\frac{x_i}{\beta^4}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2}{2}x_i^2 e^{-\alpha x_i}\right)^2}{\left(\frac{1}{\beta}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x_i^2\, e^{-\alpha x_i}\right)^2}\right) \quad (12.6)$$

$$\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) = \sum_{i=1}^{n}\left(\frac{\left(\frac{x_i}{\beta^2}\ln\left(1+\frac{x_i}{\beta}\right) + \alpha x_i^3 e^{-\alpha x_i}(1-\alpha)\right)}{\left(\frac{1}{\beta}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha+1)} + \frac{\alpha^2\beta}{2}x_i^2\, e^{-\alpha x_i}\right)^2}\right) \quad (12.7)$$

13. Applications

The Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Akaike Information Criteria Corrected (AICC), and $-2\log L$.are used to compare the goodness of fit of the fitted distribution.

The following formula can be used to determine AIC, BIC, AICC, and $-2\log L$..

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L \ and\ AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where, $k =$ number of parameters, n sample size and $-2\log L$ is the maximized value of loglikelihood function. The MLEs of the parameters for all the datasets along with their SEs (in parentheses) and the

corresponding goodness-of-fit criteria for all competing models are presented.
A basic statistical description of the dataset is given in Table 3. Figure 9, 10, and 11 indicate that Q-Q and P-P plots are suitable models for the dataset.

13.1 Figures and tables

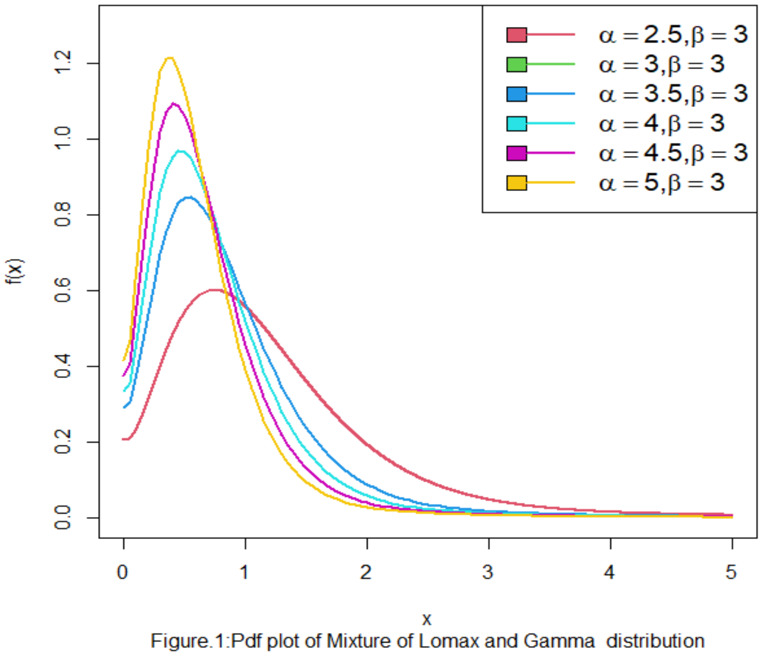

Figure 1. The MLGD distribution plot of various parameter sets. First from the: Probability density function.

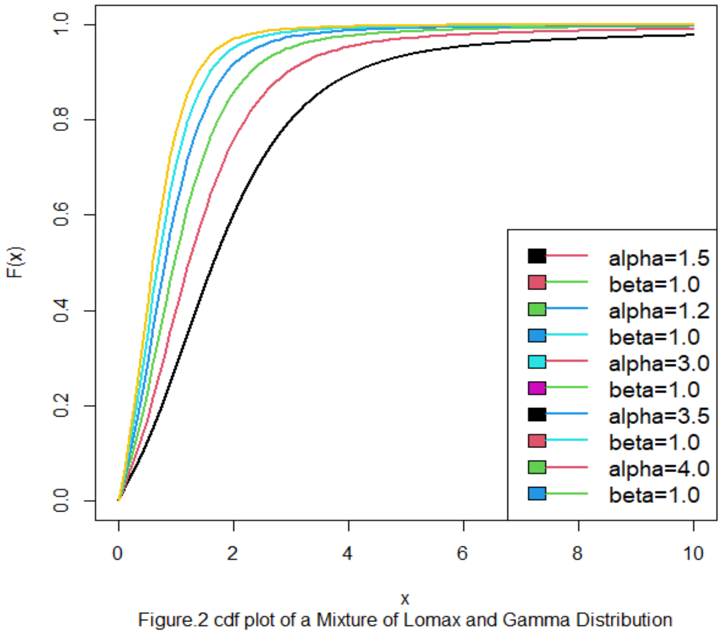

Figure 2. The MLGD distribution plot of various parameter sets. Second from the: Cumulative distribution function.

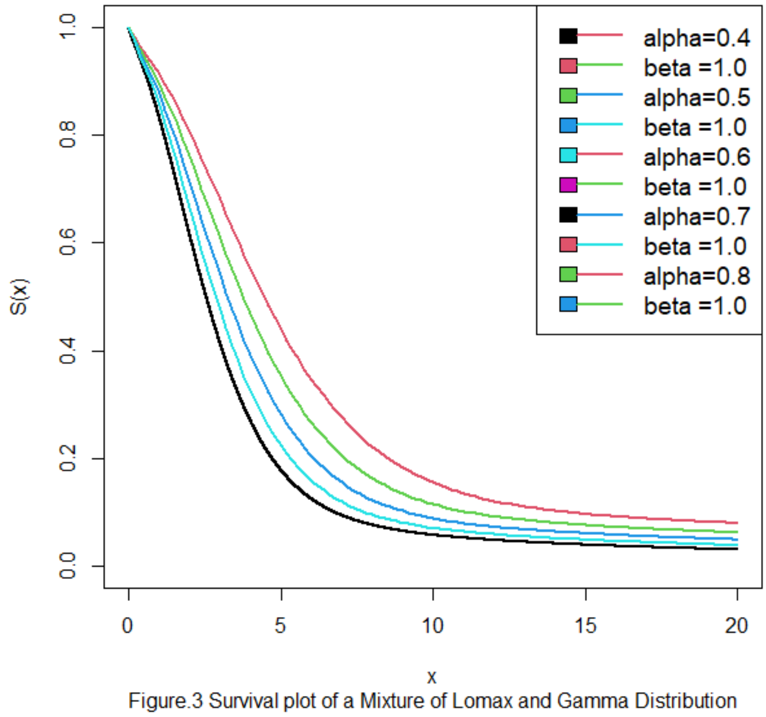

Figure 3. The MLGD distribution plot of various parameter sets. Three from the: Survival function.

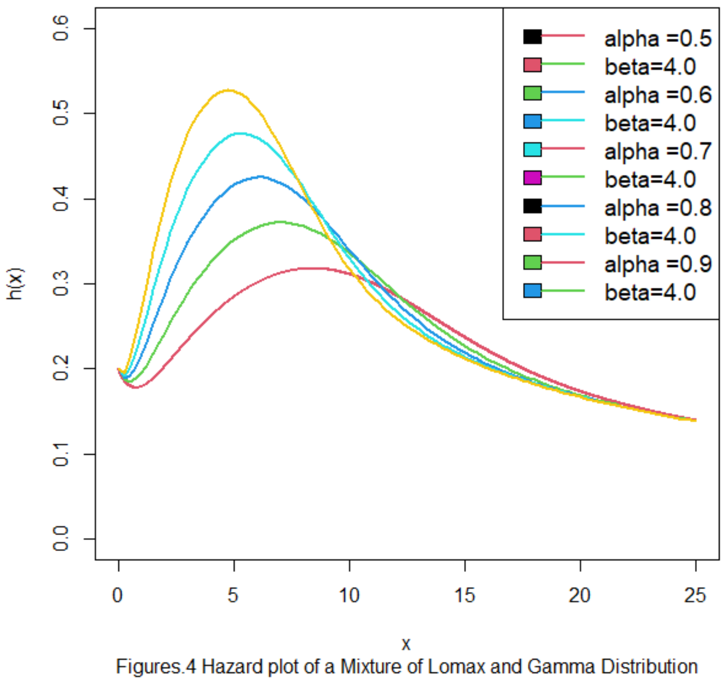

Figure 4. The MLGD distribution plot of various parameter sets. Four from the: Hazard function.

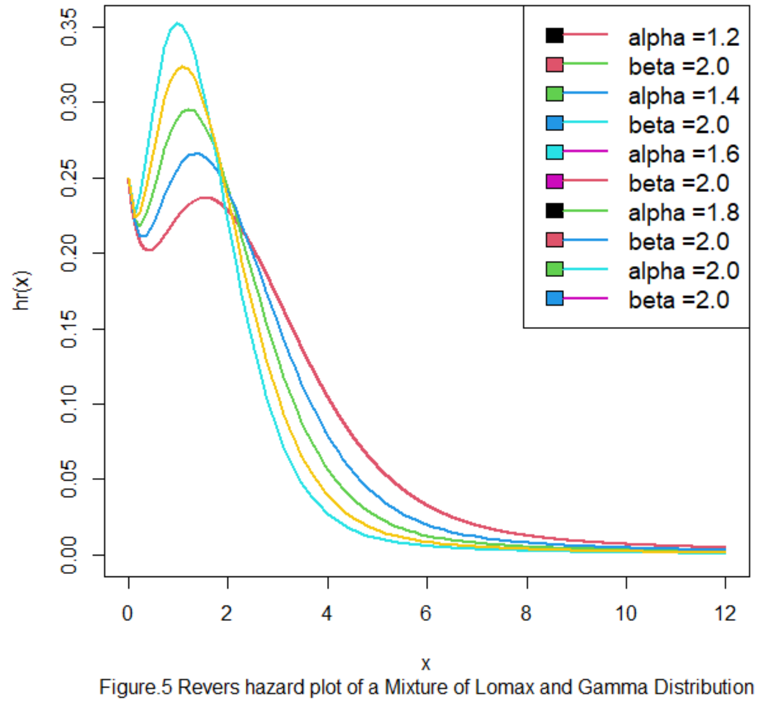

Figure 5. The MLGD distribution plot of various parameter sets. Five from the: Reverse hazard function.

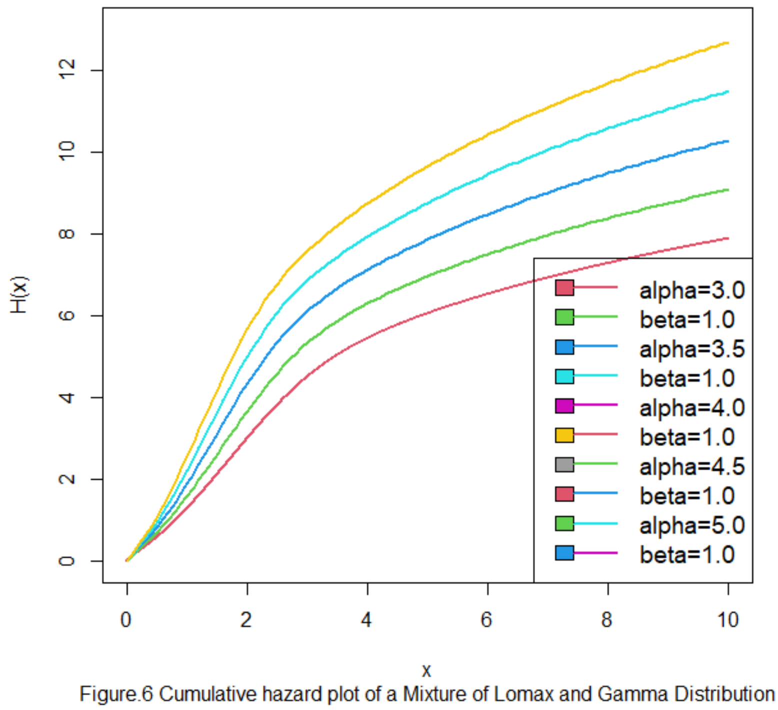

Figure 6. The MLGD distribution plot of various parameter sets. Six from the: Cumulative hazard function.

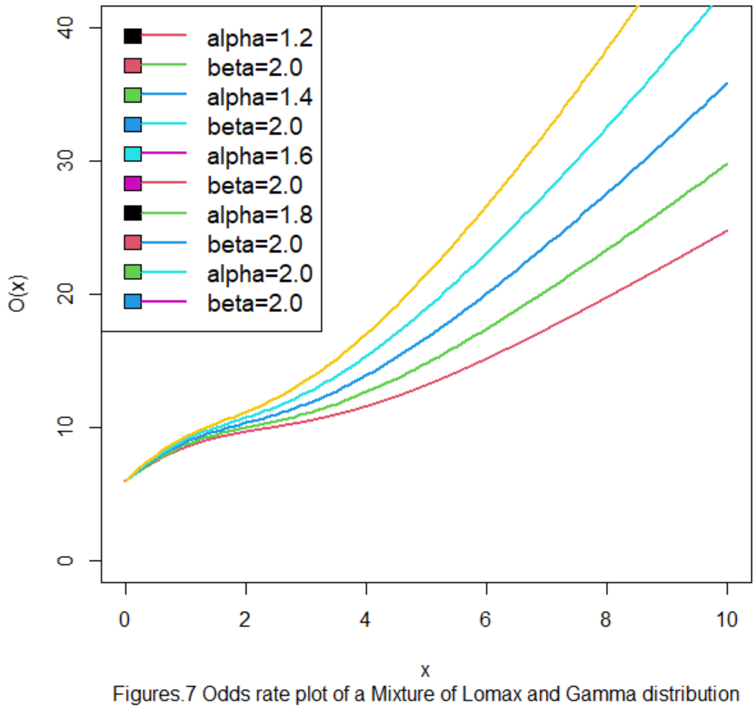

Figure 7. The MLGD distribution plot of various parameter sets. Seven from the: Odds function.

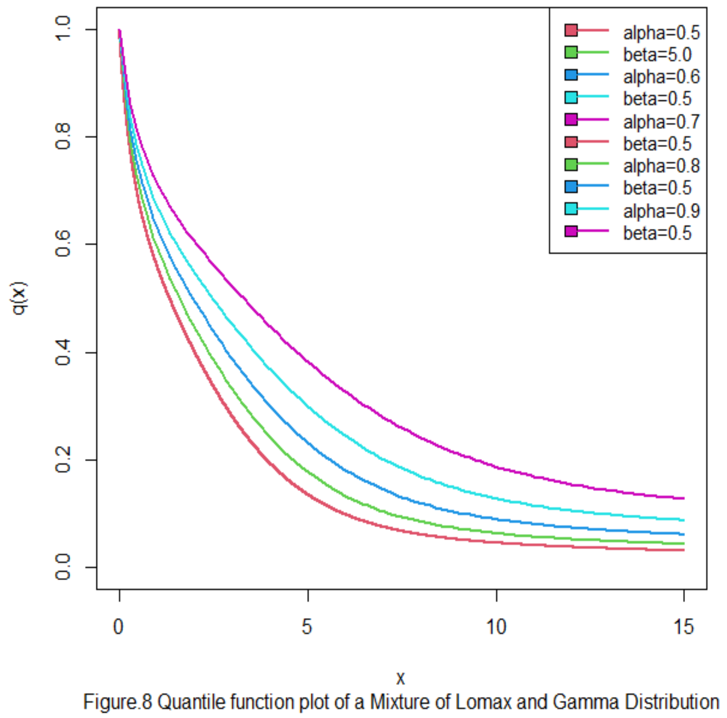

Figure 8. The MLGD distribution plot of various parameter sets. Eight from the: Quantile function.

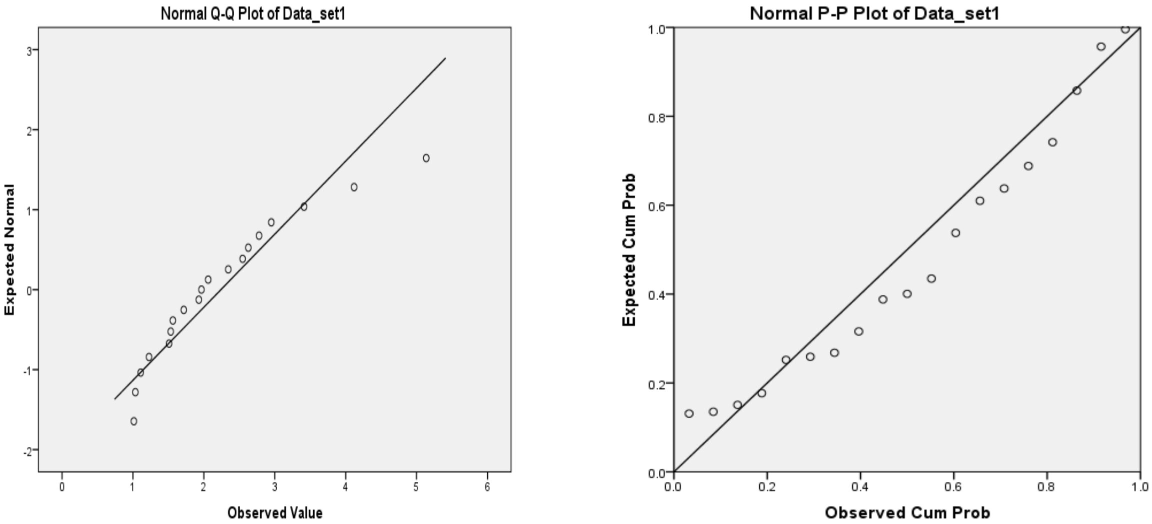

Figure 9: The density, Q-Q, P-P Plots of the cancer dataset 1

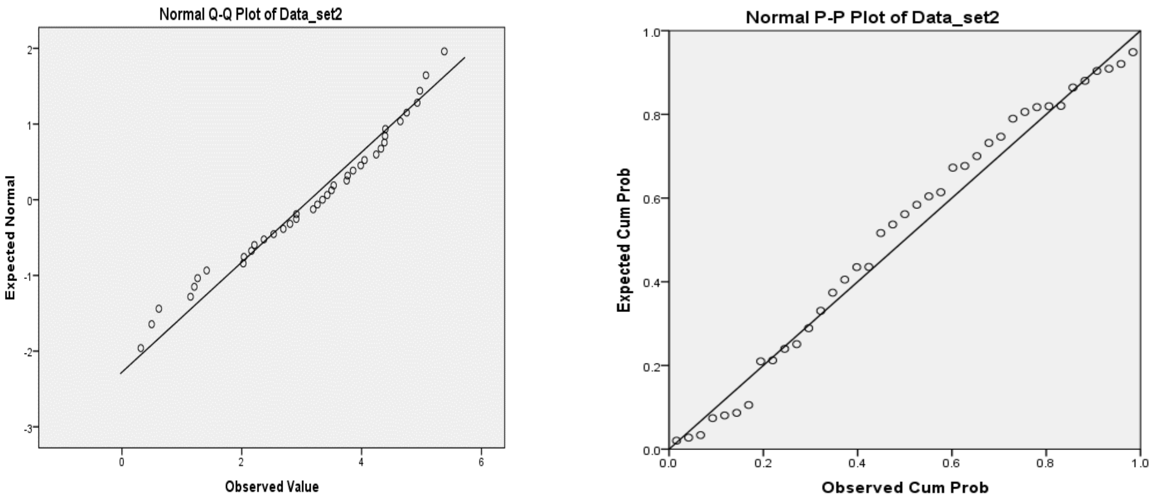

Figure 10: The density, Q-Q, P-P Plots of the cancer dataset 2

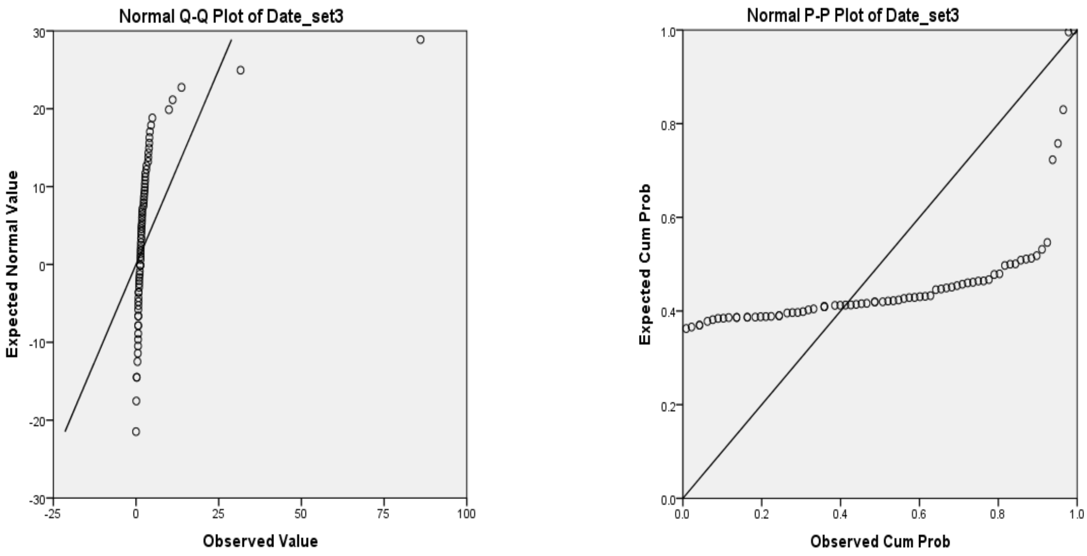

Figure 11: The density, Q-Q, P-P Plots of the cancer dataset 3

Table 1. This data includes the life expectancy (in years) of forty patients with leukemia, a blood malignancy, from one of Saudi Arabia's Ministry of Health facilities, as published in [8]. This real information is

Data Set 1	0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
	2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244	4.323
	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381	

Table 2. The data under consideration are the life times of 19 leukemia patients who were treated by a certain drug [1]. The data are

Data Set 2	1.013	1.034	1.109	1.226	1.509	1.533	1.563	1.716	1.929	1.965	2.061	2.344	2.546
	2.626	2.778	2.951	3.413	4.118	5.136							

Table 3. [10] Consider a simulated data represents the survival times (in days) of 73 patients who diagnosed with acute bone cancer, as follows

Data Set 3	0.09	0.76	1.81	1.10	3.72	0.72	2.49	1.00	0.53	0.66
	31.61	0.60	0.20	1.61	1.88	0.70	1.36	0.43	3.16	1.57
	4.93	11.07	1.63	1.39	4.54	3.12	86.01	1.92	0.92	4.04
	1.16	2.26	0.20	0.94	1.82	3.99	1.46	2.75	1.38	2.76
	1.86	2.68	1.76	0.67	1.29	1.56	2.83	0.71	1.48	2.41
	0.66	0.65	2.36	1.29	13.75	0.67	3.70	0.76	3.63	0.68
	2.65	0.95	2.30	2.57	0.61	3.39	1.56	1.29	9.94	1.67
	1.42	4.18	1.37							

Table 4. The statistical approach of the cancer patient's dataset

Data set	n	Mean	Median	Variance	Std. Deviation	Skewness	Kurtosis
1	40	3.13541	3.34800	1.894	1.376209	-.416	-.727
2	19	2.18695	1.94700	1.193	1.092260	1.264	1.529
3	73	3.7045	1.5650	110.982	10.53482	6.984	53.036

Table 5. The value of MLE's and goodness of fit criteria statistics for model selection based on cancer dataset 1

Distribution	MLE	SE	$-2\log L$	AIC	BIC	AICC
New Mixture of Lomax and Gamma distribution	$\hat{\alpha} = 9.5425$ $\hat{\beta} = 9.2663$	8.7111 3.1705	147.7154	151.7154	155.0932	152.0397
Lomax	$\hat{\alpha} = 5319.59$ $\hat{\beta} = 1667.01$	8389.06 175.4877	167.1379	171.1359	174.463	171.4602
Lindely	$\hat{\theta} = 0.2577$	0.0616	156.5028	158.5028	160.1664	158.6080
Shanker	$\hat{\theta} = 0.5497$	0.0580	144.7945	155.9545	157.6181	156.0597
Rama	$\hat{\theta} = 1.1014$	0.0805	143.3158	154.3158	147.1023	154.4210
Exponential	$\hat{\theta} = 0.3189$	0.0510	167.1353	169.1353	170.7988	169.0405
Aradhana	$\hat{\theta} = 0.7506$	0.0710	153.1793	155.1793	156.8682	155.2845
Akash	$\hat{\theta} = 0.7999$	0.0701	152.7582	154.7582	156.4471	154.8634
Ishita	$\hat{\theta} = 0.8047$	0.0642	151.6347	153.6347	155.3235	153.7399
Quasi Lindly	$\hat{\theta} = 0.6376$ $\hat{\alpha} = 0.0010$	0.1450 0.5027	149.1123	153.1123	156.4394	153.4367
Quasi Aradhana	$\hat{\theta} = 0.6376$ $\hat{\alpha} = 0.0710$	0.0817 0.1588	203.1178	207.1178	210.5049	207.4421

Table 6. The value of MLE's and goodness of fit criteria statistics for model selection based on cancer dataset 2

Distribution	MLE	SE	$-2\log L$	AIC	BIC	AICC
New Mixture of Lomax and Gamma distribution	$\hat{\alpha} = 1.3389$ $\hat{\beta} = 1091.97$	0.1773 6026.97	53.61945	57.61945	59.50835	58.3253
Lomax	$\hat{\alpha} = 2956.23$ $\hat{\beta} = 6625.11$	6851.60 390.56	68.6555	72.6555	74.5443	73.4055
Weibull	$\hat{\alpha} = 2.1959$ $\hat{\beta} = 0.4029$	0.3589 0.0435	56.0196	60.0196	62.0111	60.7692
Lindely	$\hat{\theta} = 0.7076$	0.1200	64.02158	66.02158	66.96602	66.2438
Shanker	$\hat{\theta} = 0.7124$	0.1077	63.08856	65.08856	66.033	65.3107
Rama	$\hat{\theta} = 1.3784$	0.1415	62.41991	64.41991	65.36435	64.6421
Exponential	$\hat{\theta} = 0.4463$	0.1023	68.65501	70.65501	71.59945	70.8772
Aradhana	$\hat{\theta} = 0.9855$	0.135	60.60053	62.60053	63.54497	62.8227
Akash	$\hat{\theta} = 0.0297$	0.1317	62.69158	64.69158	65.63602	64.9138
Ishita	$\hat{\theta} = 0.9975$	0.1134	62.74297	64.74297	65.68741	64.9651
Quasi Lindely	$\hat{\theta} = 0.8923$ $\hat{\alpha} = 0.0100$	0.1254 NaN	57.9066	61.9066	63.7955	62.6566
Quasi Aradhana	$\hat{\theta} = 0.8923$ $\hat{\alpha} = 0.0100$	0.1906 0.4547	84.2462	88.2462	90.1351	88.9962
Quasi Sujatha	$\hat{\theta} = 1.1737$ $\hat{\alpha} = 0.0100$	0.1244 NaN	55.9815	59.9815	61.8703	60.7315
Quasi Akash	$\hat{\theta} = 1.3713$ $\hat{\alpha} = 0.0100$	0.1296 NaN	55.7074	59.7074	61.6989	60.4574

Table 7. The value of MLE's and goodness of fit criteria statistics for model selection based on cancer dataset 3.

Distribution	MLE	SE	$-2\log L$	AIC	BIC	AICC
New Mixture of Lomax and Gamma distribution	$\hat{\alpha} = 1.6228$ $\hat{\beta} = 4.2083$	0.1508 2.6092	277.3378	281.3378	285.9187	281.5092
Lomax	$\hat{\alpha} = 2.6256$ $\hat{\beta} = 5.1372$	0.8258 2.0710	299.6024	303.6024	308.1834	303.7739
Weibull	$\hat{\alpha} = 0.7655$ $\hat{\beta} = 0.3417$	0.0567 0.0556	322.8033	326.8033	331.3842	326.9748
Aradhana	$\hat{\theta} = 0.6665$	0.0458	405.5844	407.5844	409.8748	407.6407
Ishita	$\hat{\theta} = 0.7626$	0.0449	425.5164	427.5164	429.8069	427.5727
Shanker	$\hat{\theta} = 0.5124$	0.0391	373.2109	375.2109	377.5014	375.2672
Rama	$\hat{\theta} = 0.9900$	0.0536	483.9594	485.9594	488.2499	486.0157
Exponential	$\hat{\theta} = 0.2763$	0.0323	333.7534	335.7534	338.0438	335.8097
Lindley	$\hat{\theta} = 0.4650$	0.0394	365.8631	367.8631	370.1536	367.9294
Akash	$\hat{\theta} = 0.7162$	0.0465	419.7666	421.7666	424.051	421.8230
Quasi Lindley	$\hat{\theta} = 2.6641$ $\hat{\alpha} = 2.1482$	3.1188 1.1866	339.179	343.179	347.7599	343.3505
Quasi Shanker	$\hat{\theta} = 0.4990$ $\hat{\alpha} = 0.0017$	0.0398 0.0131	383.0417	387.0417	391.6226	387.2132
Quasi Arathana	$\hat{\theta} = 0.4292$ $\hat{\alpha} = 0.4112$	0.0472 0.1399	485.3705	489.3705	493.9515	489.542
Quasi Sujath	$\hat{\theta} = 0.2827$ $\hat{\alpha} = 236.94$	0.0340 239.64	336.1475	340.1475	344.7284	340.319

14. Results and Discussion

In comparison to the new mixture of Lomax and gamma distributions, Lindely, Shanker, Rama, Exponential, Weibull, Lomax, Aradhana, Akash, Ishita, and Quasi Shanker, Quasi Lindely, Quasi Arathana, and Quasi Sujatha distributions, it is evident from table 5, 6, and 7 results that the MLG distribution has smaller AIC, BIC, and AICC values. This suggests that the new mixture distribution fits the data better. Therefore, compared to the other distributions, the new mixture of Lomax and gamma distributions (MLGD) provides a better fit.

15. Conclusions

In this paper, a new two-parameter distribution is called a MLGD, which is a new mixture of two known distributions, the Lomax and gamma distributions. Survival function and hazard function have been discussed. Some statistical properties of the moments, the moments-generating function, mean, variance, skewness, and kurtosis have been studied. A number of statistical characteristics of the proposed distribution have been derived, including order statistics, stochastic ordering, entropies, Bonferroni, and Lorenz curves, and the method of maximum likelihood estimation of the parameters has been estimated. The statistical approach of the cancer dataset was analyzed. Moreover, the derived distribution is applied to real data sets and compared with the other well-known distribution. Show that the result of the new mixture of Lomax and Gamma distributions provides a better fit than other well-known distributions.

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Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization: SAKTHIVEL, M. PANDIYAN, P. **Data curation:** SAKTHIVEL, M. PANDIYAN, P. **Formal analysis:** SAKTHIVEL, M. PANDIYAN, P. **Funding acquisition:** SAKTHIVEL, M. PANDIYAN, P. **Investigation:** SAKTHIVEL, M. PANDIYAN, P. **Methodology:** SAKTHIVEL, M. PANDIYAN, P. **Project administration:** SAKTHIVEL, M. PANDIYAN, P. **Software:** SAKTHIVEL, M. PANDIYAN, P. **Resources:** SAKTHIVEL, M. PANDIYAN**,** P. **Supervision:** SAKTHIVEL, M. PANDIYAN, P. **Validation:** SAKTHIVEL, M. PANDIYAN, P. **Visualization:** SAKTHIVEL, M. PANDIYAN, P. **Writing - original draft:** SAKTHIVEL, M. PANDIYAN, P. **Writing - review and editing:** SAKTHIVEL, M. PANDIYAN, P.

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