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ARTICLE

The Gamma-Akash probability Model for Survival Analysis of Cancer Patients[1]

Mousumi Ray*[1] and Rama Shanker[1]

[1]Department of Statistics, Assam University, Silchar, Assam, India
*Corresponding author. Email: mousumiray616@gmail.com

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Abstract

The search for a suitable probability models for the survival analysis of cancer patients are really challenging because survival times of cancer patients are stochastic in nature and are highly positively skewed. The classical well-known one parameter and two-parameter probability models rarely provide better fit to survival times of cancer patients. In this paper a compound probability model called gamma-Akash distribution, which is a compounding of gamma and Akash distribution, has been proposed for the modeling of survival times of cancer patients. Many important properties of the suggested distribution including its shape, moments (negative moments), hazard function, reversed hazard function, quantile function have been discussed. Method of maximum likelihood has been used to estimate its parameters. A simulation study has been conducted to know the consistency of maximum likelihood estimators. Two real datasets, one relating to acute bone cancer and the other relating to head and neck cancer, has been considered to examine the applicability, suitability and flexibility of the proposed distribution. The goodness of fit of the proposed distribution shows quite satisfactory fit over other considered distributions

Keywords: compounding; hazard function; reversed hazard rate function; stress-strength parameter; maximum likelihood estimation; applications.

1. Introduction

Several statistical distributions have been extensively used for the modeling and analysis of survival times (time to event) data, also known as reliability data in biomedical sciences. On comparative studies on gamma and Weibull (1951) distributions done by Shanker *et al.* (2016) shows that on some datasets relating to head and neck cancer these two classical two-parameter lifetime distributions do not provide good fit and on some datasets they perform diversely. During recent decades researchers were trying to modify Weibull distribution which would provide better fit to cancer data. As we know that the Weibull distribution is the most popular distribution for modeling survival data that properly explain the mortality and failure. Several authors have extended the Weibull distribution by adding one or more additional parameters to bring more flexibility in the shape of the distribution to accommodate the nature of the data. For example, Beta-Weibull (BW) distribution by Famoye *et al.* (2005), Kumaraswamy Weibull (Kum-W) distribution by Cordeiro *et al.* (2010), exponentiated generalized Weibull (EGW) distribution by

Cordeiro *et al.* (2013), exponentiated Kumaraswamy Weibull (EKumW) distribution by Eissa (2017), Alpha power Weibull (APW) distribution by Nassar *et al.* (2017), are some among others. Although, these two, three and four parameters extended Weibull distribution provide good fit to survival times of cancer patients, but are not quite satisfactory because, in general, cancer data are highly positively skewed. During recent decades several researchers have been trying to derive a suitable lifetime distribution to model data which are highly positively skewed, especially survival times of cancer patients. The search for highly positively skewed continuous distribution (mean is much less than the variance) has been studied by several researchers using compounding technique as the compounding always provides a highly positively skewed distributions. For instance, gamma distribution is a positively skewed distribution and compounding it with other positively skewed distribution provides highly positively skewed distribution. A compound gamma distribution arises when a random variable say X , follows gamma distribution with a shape parameter φ and scale parameter λ and the parameter λ itself behaves as a random variable with some distribution which is known as mixing distribution. There are four important one parameter positively skewed lifetime distributions namely, exponential distribution, Lindley distribution by Lindley (1958), Shanker distribution by Shanker (2015a) and Akash distribution by Shanker (2015b) for modeling and analysis of survival time of cancer patients and out of these four distributions, Akash distribution provides much better fit as compared to the other distributions. The gamma-Lindley distribution (G-LD) proposed by Abdi *et al.* (2019) which is a compound of gamma distribution with Lindley distribution of Lindley (1958) is highly positively skewed distribution. The gamma – Shanker distribution (G-SD) introduced by Ray & Shanker (2023a), which is a compound of gamma distribution with Shanker distribution of Shanker (2015a) is also highly positively skewed distribution. Further exponential-Shanker distribution (E-SD) suggested by Ray & Shanker (2023b) which is the compound of exponential distribution with Shanker distribution is also positively skewed distribution.

It has been observed by Ray & Shanker (2023a) that there are some highly positively skewed datasets where G-LD and G-SD do not provide good fit. This necessitates the search for other compound distribution. The motivations for considering the gamma-Akash distribution (G-AD), the compound of gamma and Akash distribution are as follows:

(i). Suppose, X is the lifetime of component following gamma distribution with shape parameter φ and scale parameter λ . If the sample is drawn from the population having variability in the scale parameter λ, then the variability can be well explained by assuming the distribution of λ to be Akash distribution.

(ii). In real life situation, the sustainability of the components of population differs from each other in terms of heterogeneity. The analysis of data from such populations, heterogeneity can easily be taken into consideration using compound distributions. G-LD and G-SD are the two compound distributions proposed for the analysis of such variation in the components of populations. As Akash distribution provides better fit over Lindley and Shanker distributions, it is the expectation that the G-AD would provide better fit over existing compound distributions.

(iii). In general, compound distribution is the most suited distributions for the datasets having long right tail, which have been observed in some real lifetime datasets relating to cancer datasets.

(iv). As Akash distribution performs well compared to exponential and Lindley distributions so it is hoped that G-AD would performs better over the classical gamma and Weibull distributions as well as other two-parameters distributions.

The whole paper is divided into five sections. The section one is introductory in nature. Materials and Methods are given in section two. Statistical properties of the proposed probability model are given in section three . Section four contains results and discussion. The conclusion of the whole paper is given in section five.

2. Materials and Methods

The method of deriving the compound of positively skewed distribution is as follow: Let X be a continuous random variable having positively skewed pdf $g(x|\lambda),\lambda$ may be a vector of parameter. Let λ is also a random variable having a positively skewed pdf $h(\lambda|\beta)$, then the unconditional (marginal) distribution of X can be obtained as

$$f(x;\beta)=\int g(x|\lambda)h(\lambda|\beta)d\lambda$$

This is known as compound distribution and the process of obtaining compound distribution is known as compounding. Following this approach of deriving positively skewed distribution, Abdi *et al.* (2019) and Ray & Shanker (2023a) obtained G-LD and G-SD, where the gamma distribution has been compounded with Lindley distribution and Shanker distribution, respectively.

The G-LD and the G-SD for $x>0,\varphi>0,\omega>0$ are defined by its probability density function (pdf) and cumulative density function (cdf) as follows

$$f_{G-LD}(x;\varphi,\omega)=\frac{\varphi\omega^2(1+\varphi+\omega+x)x^{\varphi-1}}{(\omega+1)(\omega+x)^{\varphi+2}} \tag{1}$$

$$F_{G-LD}(x;\varphi,\omega)=\frac{x^{\varphi}\left[(\omega+1)x+(1+\varphi+\omega)\omega\right]}{(\omega+1)(\omega+x)^{\varphi+1}} \tag{2}$$

$$f_{G-SD}(x;\varphi,\omega)=\frac{\varphi\omega^2(1+\varphi+\omega x+\omega^2)x^{\varphi-1}}{(1+\omega^2)(\omega+x)^{2+\varphi}} \tag{3}$$

$$F_{G-SD}(x;\varphi,\omega)=\frac{x^{\varphi}\left[x(1+\omega^2)+(1+\varphi+\omega^2)\omega\right]}{(1+\omega^2)(\omega+x)^{1+\varphi}} \tag{4}$$

Akash distribution is defined by its pdf and cdf

$$f_{AD}(x;\omega)=\frac{\omega^3(1+x^2)e^{-\omega x}}{(\omega^2+2)} \tag{5}$$

$$F_{AD}(x;\omega)=1-\left[1+\frac{\omega x(\omega x+2)}{\omega^2+2}\right]e^{-\omega x} \tag{6}$$

The gamma-Akash distribution (G-AD) is the compound of gamma distribution with the Akash distribution. The pdf and the cdf of G-AD, following the above approach of compounding, are obtained as

$$f_{G-AD}(x;\varphi,\omega)=\frac{\varphi\omega^3\left[(\omega+x)^2+(\varphi+2)(\varphi+1)\right]x^{\varphi-1}}{(\omega^2+2)(\omega+x)^{\varphi+3}};x>0,\varphi>0,\omega>0 \tag{7}$$

$$F_{G-AD}\left(x;\varphi,\omega\right)=\frac{x^{\varphi}\begin{bmatrix}\left(\omega^{2}+2\right)x^{2}+2\left(\omega^{2}+\varphi+2\right)\omega x\\+\left(\omega^{2}+\varphi^{2}+3\varphi+2\right)\omega^{2}\end{bmatrix}}{\left(\omega^{2}+2\right)\left(\omega+x\right)^{\varphi+2}};x>0,\varphi>0,\omega>0 \quad (8)$$

The pdf in equation (7) is known as G-AD. The behaviours of pdf and cdf of G-AD have been studied for various combinations of parameters. Figure 1 and 2 shows the pdf and cdf of G-AD for selected values of parameters. The G-AD shows the tendency to accommodate right tail and for particular values of parameters, the tail approach to zero at a faster rate. This means that G-AD would provide better fit appropriately to those datasets where there is an extended right tail or the right tail approaches to zero at a faster rate. Such datasets are quite prevalent in the biomedical sciences relating to survival times of cancer patients which has been discussed in Klakattawi (2022).

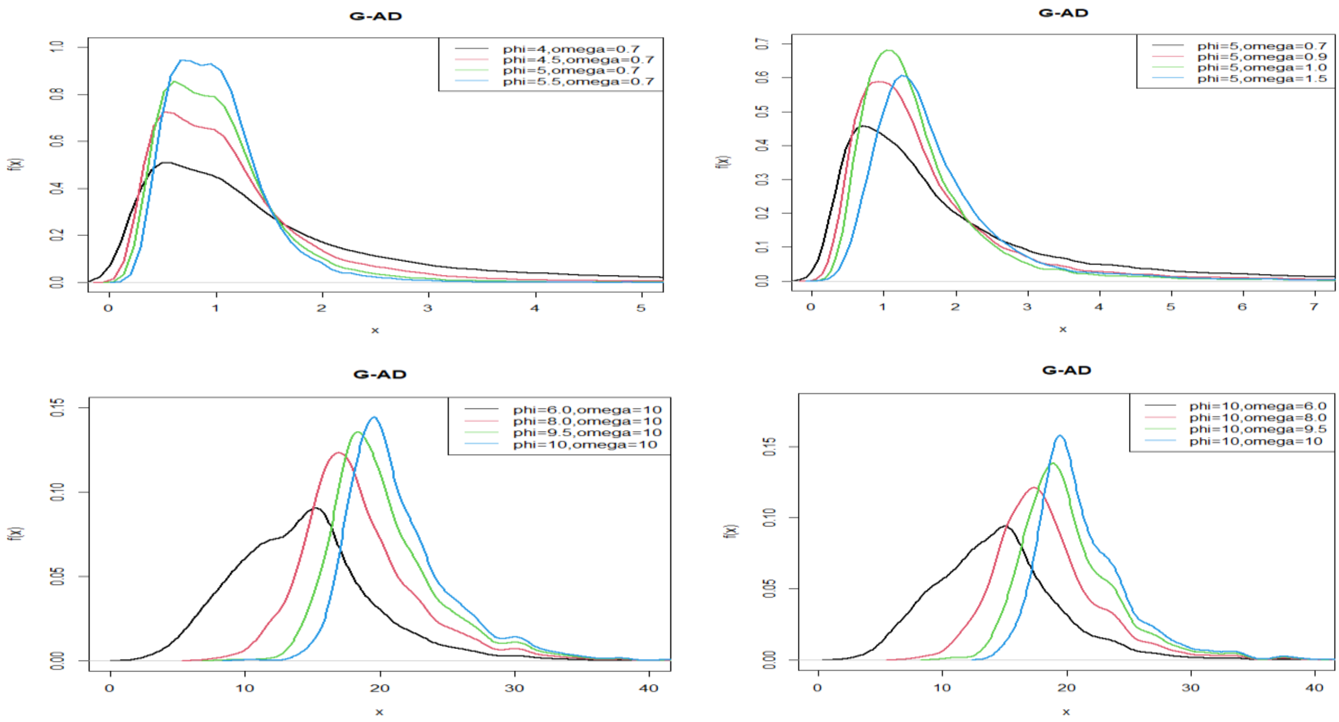

Figure 1. Probability density function (pdf) plots of G-AD.

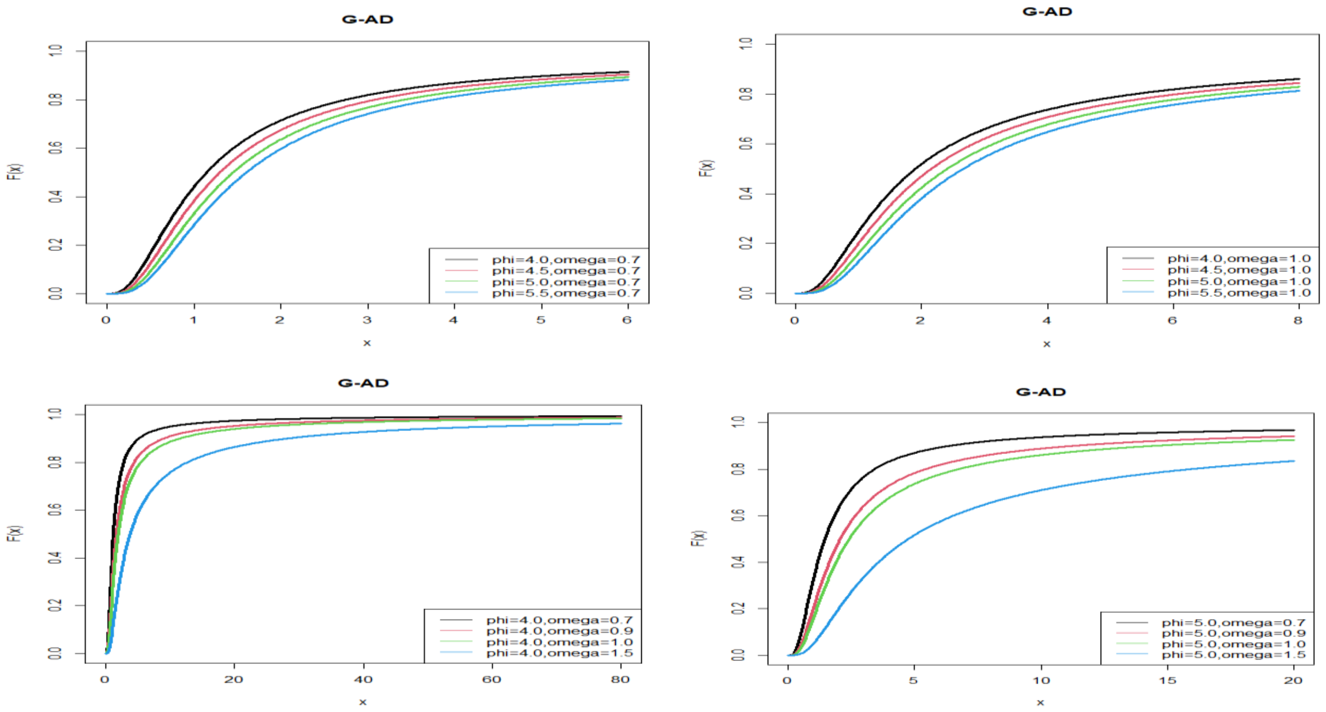

Figure 2. Cumulative density function (cdf) plots of G-AD.

3. Statistical properties of G-AD

In this section we presnted several interesting statistical properties of G-AD.

3.1 Nature of G-AD

Theorem 1: The pdf of G-AD is decreasing for $\varphi \le 1$

Proof: We have,

$$f(x;\varphi,\omega)=\frac{\varphi\omega^3\left[(\omega+x)^2+(\varphi+2)(\varphi+1)\right]x^{\varphi-1}}{(\omega^2+2)(\omega+x)^{\varphi+3}};x>0,\varphi>0,\omega>0$$

$$\log f(x;\varphi,\omega)=\log\left[(\omega+x)^2+(\varphi+2)(\varphi+1)\right]+(\varphi-1)\log(x)-(\varphi+3)\log(\omega+x)+C$$

where C is a constant. We have

$$\frac{d}{dx}\log f(x;\varphi,\omega)=\frac{\varphi-1}{x}-\left[\frac{(\varphi+1)\left\{(\omega+x)^2+(\varphi+2)(\varphi+3)\right\}}{(\omega+x)\left\{(\omega+x)^2+(\varphi+1)(\varphi+2)\right\}}\right]$$

For $\varphi \le 1$, $\frac{d}{dx}\log f(x;\varphi,\omega)<0$ and this means that $f(x)$ is decreasing for all x.

3.1.1 Hazard function and Reversed hazard function

The hazard rate function and the reverse hazard rate function are two important functions of a distribution. The reliability (survival) function of G-AD is given by

$$R(x;\varphi,\omega)=1-F(x;\varphi,\omega)=\frac{(\omega^2+2)(\omega+x)^{\varphi+2}-x^{\varphi}\left[\begin{array}{l}(\omega^2+2)x^2+2(\omega^2+\varphi+2)\omega x\\+(\omega^2+\varphi^2+3\varphi+2)\omega^2\end{array}\right]}{(\omega^2+2)(\omega+x)^{\varphi+2}} \tag{9}$$

The corresponding hazard function and reversed hazard function of G-AD are given by

$$h(x;\varphi,\omega)=\frac{f(x;\varphi,\omega)}{R(x;\varphi,\omega)}$$

$$=\frac{\varphi\omega^3x^{\varphi-1}\left[(\omega+x)^2+(\varphi+1)(\varphi+2)\right]}{(\omega+x)\left[(\omega^2+2)(\omega+x)^{\omega+2}-x^{\varphi}\left\{(\omega^2+2)x^2+2(\omega^2+\varphi+2)\omega x+(\omega^2+\varphi^2+3\varphi+2)\omega^2\right\}\right]} \tag{10}$$

$$r(x;\varphi,\omega)=\frac{f(x;\varphi,\omega)}{F(x;\varphi,\omega)}$$

$$=\frac{\varphi\omega^3\left[(\omega+x)^2+(\varphi+1)(\varphi+2)\right]}{x(\omega+x)\left[(\omega^2+2)x^2+2(\omega^2+\varphi+2)\omega x+(\omega^2+\varphi^2+3\varphi+2)\omega^2\right]} \tag{11}$$

Figures 3 and 4 shows the hazard function and reversed hazard function of G-AD for selected values of parameters.

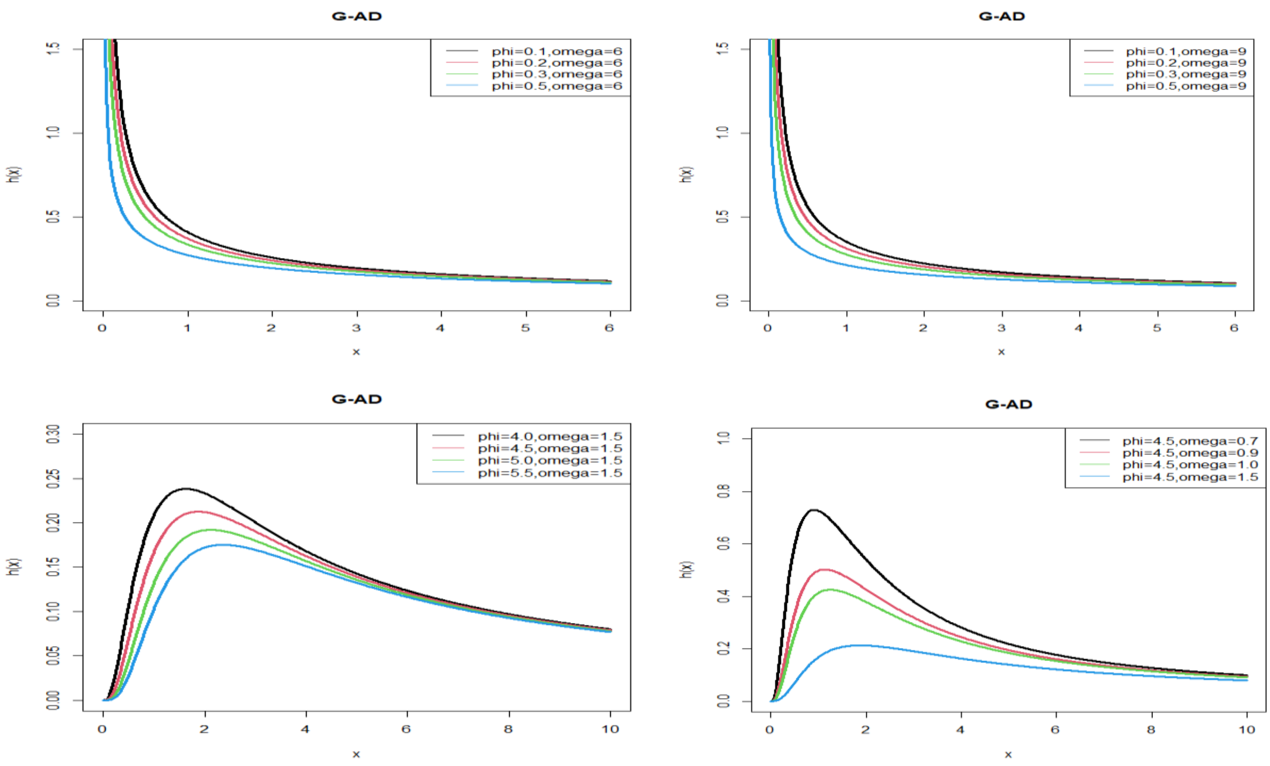

Figure 3. Hazard function of G-AD.

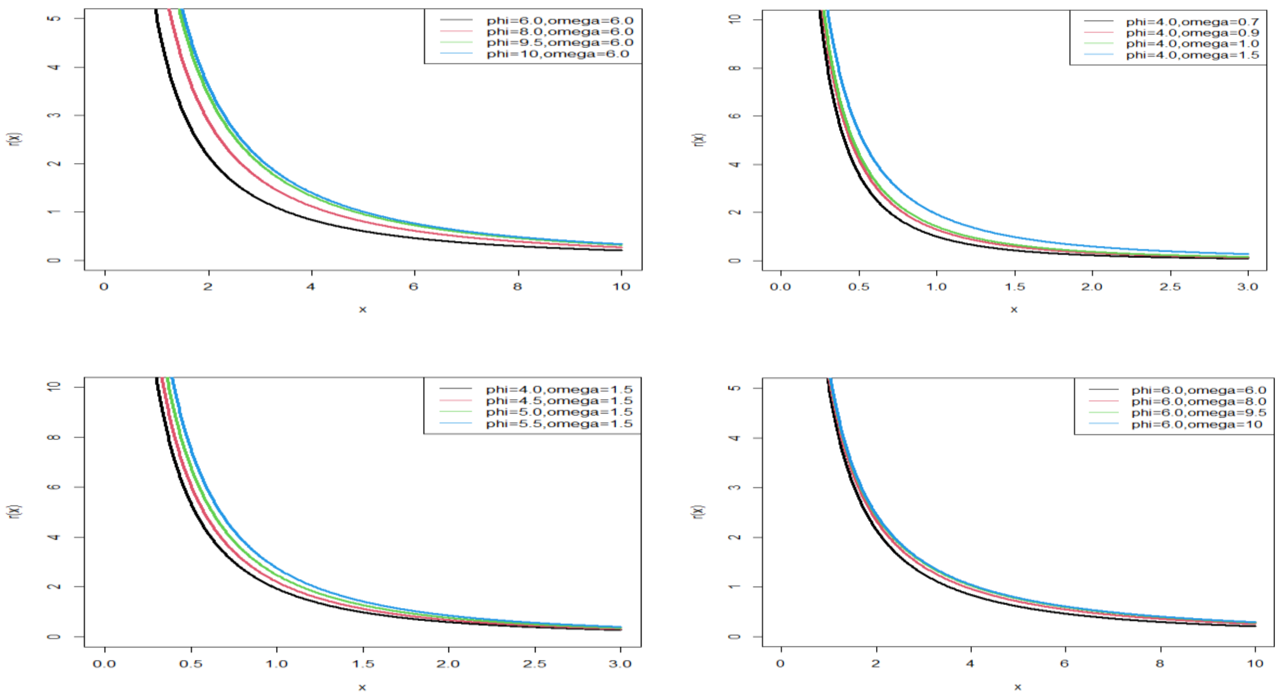

Figure 4. Reverse hazard function of G-AD.

Theorem 2: For $\varphi \le 1$, the hazard function of the G-AD is decreasing and for $\varphi > 1$ it is unimodal.

Proof: We have

$$f\left(x;\varphi,\omega\right)=\frac{\varphi\omega^{3}\left[\left(\omega+x\right)^{2}+\left(\varphi+2\right)\left(\varphi+1\right)\right]x^{\varphi-1}}{\left(\omega^{2}+2\right)\left(\omega+x\right)^{\varphi+3}} \quad \text{and}$$

(12)

$$f'\left(x\right)=\frac{\varphi\omega^{3}x^{\varphi-2}\begin{bmatrix}\left(\varphi-1\right)\left(\omega+x\right)^{\varphi+3}\left\{\left(\omega+x\right)^{2}+\left(\varphi+1\right)\left(\varphi+2\right)\right\}\\ -x\left\{\left(\omega+x\right)^{2}+\left(\varphi+1\right)\left(\varphi+2\right)\left(\varphi+3\right)\left(\omega+x\right)^{\varphi+2}-2\left(\omega+x\right)^{\varphi+4}\right\}\end{bmatrix}}{\left(\omega^{2}+2\right)\left(\omega+x\right)^{2\alpha+6}}$$

Now, suppose that

$$\xi\left(x\right)=-\frac{f'\left(x\right)}{f\left(x\right)}=-\frac{\left(\varphi-1\right)}{x}+\frac{1}{\left(\omega+x\right)^{\varphi+1}\left[\left(\omega+x\right)^{2}+\left(\varphi+1\right)\left(\varphi+2\right)\right]}+\frac{\left(\varphi+1\right)\left(\varphi+2\right)\left(\varphi+3\right)}{\left(\omega+x\right)\left[\left(\omega+x\right)^{2}+\left(\varphi+1\right)\left(\varphi+2\right)\right]}-\frac{2\left(\omega+x\right)}{\left[\left(\omega+x\right)^{2}+\left(\varphi+1\right)\left(\varphi+2\right)\right]}$$

(13)

$$\xi'(x)=\frac{(\varphi-1)}{x^2}-\frac{\left[(\varphi+3)(\omega+x)^2+(\varphi+1)^2(\varphi+2)\right]}{(\omega+x)^{\varphi+6}\left\{(\omega+x)^2+(\varphi+1)(\varphi+2)\right\}^2}-\frac{(\varphi+1)(\varphi+2)(\varphi+3)\left[3(\omega+x)^2+(\varphi+1)(\varphi+2)\right]}{(\omega+x)^2\left\{(\omega+x)^2+(\varphi+1)(\varphi+2)\right\}^2}$$
$$-\frac{2\left\{(\varphi+1)(\varphi+2)-2(\omega+x)^2\right\}}{\left\{(\omega+x)^2+(\varphi+1)(\varphi+2)\right\}^2} \qquad (14)$$

Using the lemma given by Glaser (1980), which states that if $-f'(x)/f(x)<0$, then the hazard function is decreasing , it is quite obvious that for $\varphi\le 1$, $\xi'(x)<0$ and for $\varphi>1$, $\xi'(x)<0$ and thus hazard function of G-AD is decreasing.

Theorem 3: The G-AD has decreasing reverse hazard function.
Proof: We have,

$$r(x)=\frac{\varphi\omega^3\left[(\omega+x)^2+(\varphi+1)(\varphi+2)\right]}{x(\omega+x)\left[(\omega^2+2)x^2+2(\omega^2+\varphi+2)\omega x+(\omega^2+\varphi^2+3\varphi+2)\omega^2\right]} \qquad (15)$$

This gives

$$\frac{d}{dx}\log r(x)$$
$$=\frac{-\varphi\omega\left\{2(\varphi^2+\omega^2)^2+10\varphi+16-x^2\right\}-4\left\{x(\varphi^2+\omega^2+3\varphi+2)+\omega(\omega^2+2)\right\}}{\left\{(\omega+x)^2+(\varphi+1)(\varphi+2)\right\}\left\{(\omega^2+2)x^2+2(\omega^2+\varphi+2)\omega x+(\omega^2+\varphi^2+3\varphi+2)\omega^2\right\}}-\frac{1}{x}-\frac{1}{(\omega+x)}<0$$

for all φ,ω .This proves the theorem.

3.1.2 Quantiles and Moments of G-AD

The pth quantiles x_p of G-AD is defined by $F(x_p)=p$,is the root of the equation

$$\frac{{x_p}^{\varphi}\left[(\omega^2+2){x_p}^2+2(\omega^2+\varphi+2)\omega x_p+(\omega^2+\varphi^2+3\varphi+2)\omega^2\right]}{(\omega^2+2)(\omega+x_p)^{\varphi+2}}=p \qquad (16)$$

This gives

$$\frac{\left[(\omega^2+2){x_p}^2+2(\omega^2+\varphi+2)\omega x_p+(\omega^2+\varphi^2+3\varphi+2)\omega^2\right]}{p(\omega^2+2)(\omega+x_p)\left(1+\frac{\omega}{x_p}\right)^{\varphi+1}}=x_p \qquad (17)$$

It should be noted that this x_p may be used to generate G-AD random variates. Further, the median of G-AD can be obtained from above equation by taking $p=\frac{1}{2}$.

The moments of G-AD can be obtained as follows:

If $X \sim$ G-AD(φ,ω) then,

$$E(X)=E\left(E(X\mid\lambda)\right)=E\left(\frac{\varphi}{\lambda}\right)=\varphi E\left(\frac{1}{\lambda}\right)=\infty.$$

Thus, in general, $E\left(X^r\right)=\infty$ for $r\geq 1$.

According to Abdi *et al.* (2019) this means that all moments of G-AD are infinite and hence G-AD has no mean. As G-AD has no mean, if we take a sample $\left(X_1,X_2,...,X_n\right)$ from G-AD, then mean $\bar{X}$ does not tend to a particular value. Since G-AD has no raw and central moments, we have to derive inverse (negative) moments. Negative moments are useful in several life applications, such as life testing problems and estimation purpose. The negative moments for G-AD can be obtained as follows:

The r^{th} negative moment (about origin) $\mu_{(-r)}'$,of the G-AD is given by,

$$\mu_{(-r)}'=E\left(X^{-r}\right)=E\left(E\left(X^{-r}\mid\lambda\right)\right)$$

$$=\int_0^\infty\left[\int_0^\infty x^{-r}\,\frac{\lambda^{\varphi}x^{\varphi-1}e^{-\lambda x}}{\Gamma(\varphi)}\,dx\right]\frac{\omega^3}{\omega^2+2}\left(1+\lambda^2\right)e^{-\omega\lambda}d\lambda$$

$$=\frac{\Gamma(\varphi-r)}{\Gamma(\varphi)}\cdot\frac{r!\left[\omega^2+(r+2)(r+1)\right]}{\omega^r\left(\omega^2+2\right)};r=1,2,3,... \tag{18}$$

Thus, for $r=1,2,3,4$ we have

$$\mu_{(-1)}'=E\left(\frac{1}{X}\right)=\frac{\left(\omega^2+6\right)}{\omega\left(\omega^2+2\right)(\varphi-1)},\varphi>1 \tag{19}$$

$$\mu_{(-2)}'=E\left(\frac{1}{X^2}\right)=\frac{2\left(\omega^2+12\right)}{\omega^2\left(\omega^2+2\right)(\varphi-1)(\varphi-2)},\varphi>2 \tag{20}$$

$$\mu_{(-3)}'=E\left(\frac{1}{X^3}\right)=\frac{6\left(\omega^2+20\right)}{\omega^3\left(\omega^2+2\right)(\varphi-1)(\varphi-2)(\varphi-3)},\varphi>3 \tag{21}$$

$$\mu_{(-4)}'=E\left(\frac{1}{X^4}\right)=\frac{24\left(\omega^2+30\right)}{\omega^4\left(\omega^2+2\right)(\varphi-1)(\varphi-2)(\varphi-3)(\varphi-4)},\varphi>4 \tag{22}$$

It is obvious from the above expressions for negative moments that negative moments are not defined for $\varphi\leq 1$.

3.1.3 Extreme order statistics

Let, $X_{1:n},...,X_{n:n}$ be the order statistics of a random sample of size n from the G-AD(φ,ω) distribution with distribution function $F(x)$. The cdf of the minimum order statistic $X_{1:n}$ is given by

$$F_{X_{1:n}}\left(x\right)=1-\left[1-F\left(x\right)\right]^{n}$$
$$=1-\left[\frac{\left(\omega^{2}+2\right)\left(\omega+x\right)^{2+\varphi}-x^{\varphi}\left[\begin{array}{l}\left(\omega^{2}+2\right)x^{2}+2\left(\omega^{2}+\varphi+2\right)\omega x\\+\left(\omega^{2}+\varphi^{2}+3\varphi+2\right)\omega^{2}\end{array}\right]}{\left(\omega^{2}+2\right)\left(\omega+x\right)^{2+\varphi}}\right]^{n}$$

(23)

The cdf of the maximum order statistic $X_{1:n}$ is given by

$$F_{X_{n:n}}\left(x\right)=\left[F\left(x\right)\right]^{n}=\left\{\frac{x^{\varphi}\left[\begin{array}{r}\left(\omega^{2}+2\right)x^{2}+2\left(\omega^{2}+\varphi+2\right)\omega x\\+\left(\omega^{2}+\varphi^{2}+3\varphi+2\right)\omega^{2}\end{array}\right]}{\left(\omega^{2}+2\right)\left(\omega+x\right)^{\varphi+2}}\right\}^{n}$$

(24)

3.1.4 Stochastic Orderings

In probability theory and Statistics, a stochastic order quantifies the concept of one random variable being "bigger" than other. In many problems, it becomes necessary to compare two lifetime distributions with reference to some of their characteristics. Stochastic orders provide the necessary tools in such case.

A random variable X is said to be smaller than a random variable Y in the

i. Stochastic order $\left(X\prec_{st}Y\right)$ if $F_X\left(x\right)\geq F_Y\left(y\right)$ for all x

ii. Hazard rate order $\left(X\prec_{hr}Y\right)$ if $h_X\left(x\right)\geq h_Y\left(y\right)$ for all x

iii. Mean residual life order $\left(X\prec_{mrl}Y\right)$ if $m_X\left(x\right)\geq m_Y\left(y\right)$ for all x

iv. Likelihood ratio order $\left(X\prec_{lr}Y\right)$ if $\frac{f_X\left(x\right)}{f_Y\left(Y\right)}$ decrease in x

iv. Likelihood ratio order $\left(X\prec_{lr}Y\right)$ if $\frac{f_X\left(x\right)}{f_Y\left(Y\right)}$ decrease in x

The following results due to Shaked & Shantikumar (1994) are well known for establishing stochastic ordering of distributions:

$$\begin{array}{c}X\prec_{lr}Y\Rightarrow X\prec_{hr}Y\Rightarrow X\prec_{mrl}Y\\\Downarrow\\X\prec_{st}Y\end{array}$$

Theorem 4: Let $X_1 \sim$ G-AD$\left(\varphi_1,\omega_1\right)$ and $X_2 \sim$ G-AD$\left(\varphi_2,\omega_2\right)$.If $\varphi_1=\varphi_2=\varphi$ and $\omega_1\leq\omega_2$ if $\omega_1=\omega_2=\omega\geq1$ with $\varphi_1\leq\varphi_2$, then $X_1\prec_{lr}X_2\Rightarrow X_1\prec_{hr}X_2\Rightarrow X_1\prec_{st}X_2$.

Proof: We have

$$\frac{f_{X_1}(x)}{f_{X_2}(x)}=\frac{\varphi_1\omega_1^3\left[(\omega_1+x)^2+(\varphi_1+2)(\varphi_1+1)\right](\omega_2^2+2)(\omega_2+x)^{\varphi_2+3}}{\varphi_2\omega_2^3\left[(\omega_2+x)^2+(\varphi_2+2)(\varphi_2+1)\right](\omega_1^2+2)(\omega_1+x)^{\varphi_1+3}}x^{\varphi_1-\varphi_2} \tag{25}$$

Case I: For $\varphi_1=\varphi_2=\varphi$, we get

$$G_1(x)=\frac{\omega_1^3\left[(\omega_1+x)^2+(\varphi+2)(\varphi+1)\right](\omega_2^2+2)}{\omega_2^3\left[(\omega_2+x)^2+(\varphi+2)(\varphi+1)\right](\omega_1^2+2)}\left(\frac{\omega_2+x}{\omega_1+x}\right)^{\varphi+3}$$

$$\frac{d\log G_1(x)}{dx}=\left(\frac{\varphi+3}{\omega_2+x}-\frac{2(\omega_2+x)}{(\omega_2+x)^2+(\varphi+1)(\varphi+2)}\right)-\left(\frac{\varphi+3}{\omega_1+x}-\frac{2(\omega_1+x)}{(\omega_1+x)^2+(\varphi+1)(\varphi+2)}\right)$$

$$=Q(\omega_2)-Q(\omega_1) \tag{26}$$

Where

$$Q(\omega)=\left(\frac{\varphi+3}{\omega+x}-\frac{2(\omega+x)}{(\omega+x)^2+(\varphi+1)(\varphi+2)}\right)$$

$$\frac{d}{d\omega}Q(\omega)=\frac{-(\varphi+3)}{(\omega+x)^2}-\frac{2\left\{(\varphi+2)(\varphi+1)-(\omega+x)^2\right\}}{\left\{(\omega+x)^2+(\varphi+1)(\varphi+2)\right\}^2}<0 \tag{27}$$

The X_1 is stochastically smaller than X_2 with respect to the likelihood ratio for $\varphi_1=\varphi_2=\varphi$ provided $\omega_1\le\omega_2$.

Case II: For $\omega_1=\omega_2=\omega\ge 1$, we get

$$G_2(x)=\frac{\varphi_1\left[(\omega+x)^2+(\varphi_1+1)(\varphi_1+2)\right]}{\varphi_2\left[(\omega+x)^2+(\varphi_2+1)(\varphi_2+2)\right]}\left(\frac{x}{\omega+x}\right)^{\varphi_1-\varphi_2} \tag{28}$$

$$\frac{d\log G_2(x)}{dx}=\left(\frac{2(\omega+x)}{(\omega+x)^2+(\varphi_1+1)(\varphi_1+2)}+\frac{\varphi_1}{x}-\frac{\varphi_1}{\omega+x}\right)-\left(\frac{2(\omega+x)}{(\omega+x)^2+(\varphi_2+1)(\varphi_2+2)}+\frac{\varphi_2}{x}-\frac{\varphi_2}{\omega+x}\right)$$

$$=S(\varphi_1)-S(\varphi_2) \tag{29}$$

Where, $$S(\varphi)=\left(\frac{2(\omega+x)}{(\omega+x)^2+(\varphi+1)(\varphi+2)}+\frac{\varphi}{x}-\frac{\varphi}{\omega+x}\right)$$

$$\frac{d}{d\varphi}S(\varphi)=\frac{-2(\omega+x)(2\varphi+3)}{\left\{(\omega+x)^2+(\varphi+1)(\varphi+2)\right\}^2}+\frac{1}{x}-\frac{1}{\omega+x}>0 \text{ for } \omega\ge 1$$

Thus, for $\varphi_1\le\varphi_2$, $\frac{d\log G_2(x)}{dx}<0$. The X_1 is stochastically smaller than X_2 with respect to the likelihood ratio for $\omega_1=\omega_2=\omega\ge 1$ provided $\varphi_1\le\varphi_2$.

3.2 Estimation of parameters

Let $(x_1,x_2,...,x_n)$ be the observed values of a random sample $(X_1,X_2,...,X_n)$ from the G-AD. Then the log-likelihood function is given by

$$L(\varphi,\omega)=\left(\frac{\varphi\omega^3}{\omega^2+2}\right)^n \frac{\prod_{i=1}^{n}\left[(\omega+x_i)^2+(\varphi+1)(\varphi+2)\right]\left(\prod_{i=1}^{n}x_i\right)^{\varphi-1}}{\prod_{i=1}^{n}(\omega+x_i)^{\varphi+3}} \tag{30}$$

The log-likelihood function of G-AD is thus obtained as

$$\ln L(\varphi,\omega)=n\ln\varphi+3n\ln\omega-n\ln(\omega^2+2)+\sum_{i=1}^{n}\ln\left[(\omega+x_i)^2+(\varphi+1)(\varphi+2)\right]$$
$$+(\varphi-1)\sum_{i=1}^{n}\ln(x_i)-(\varphi+3)\sum_{i=1}^{n}\ln(\omega+x_i) \tag{31}$$

The maximum likelihood estimators of φ and ω, say $\hat{\varphi}$ and $\hat{\omega}$ are the simultaneous solutions of the following log likelihood equations

$$\frac{\partial\ln L(\varphi,\omega)}{\partial\varphi}=\frac{n}{\varphi}+\sum_{i=1}^{n}\frac{(2\varphi+3)}{(\omega+x_i)^2+(\varphi+1)(\varphi+2)}+\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\ln(\omega+x_i)=0 \tag{32}$$

$$\frac{\partial\ln L(\varphi,\omega)}{\partial\omega}=\frac{3n}{\omega}-\frac{2\omega n}{(\omega^2+2)}+\sum_{i=1}^{n}\frac{2(\omega+x_i)}{\left\{(\omega+x_i)^2+(\varphi+1)(\varphi+2)\right\}}-(\varphi+3)\sum_{i=1}^{n}\frac{1}{(\omega+x_i)}=0 \tag{33}$$

It is very difficult to solve these two log-likelihood equations directly, so we will use Fisher's scoring method. We have

$$\frac{\partial^2\ln L(\varphi,\omega)}{\partial\varphi^2}=\frac{-n}{\varphi^2}+\sum_{i=1}^{n}\frac{2\left[(\omega+x_i)(\omega-2\varphi+x-3)+(\varphi+1)(\varphi+2)\right]}{\left\{(\omega+x_i)^2+(\varphi+1)(\varphi+2)\right\}^2}$$

$$\frac{\partial^2\ln L(\varphi,\omega)}{\partial\varphi\partial\omega}=-\sum_{i=1}^{n}\frac{2(\omega+x_i)(2\varphi+3)}{\left\{(\omega+x_i)^2+(\varphi+1)(\varphi+2)\right\}^2}-\sum_{i=1}^{n}\frac{1}{\omega+x_i}=\frac{\partial^2\ln L(\varphi,\omega)}{\partial\omega\partial\varphi}$$

$$\frac{\partial^2\ln L(\varphi,\omega)}{\partial\omega^2}=\frac{-3n}{\omega^2}-\frac{2n\left[(\omega^2+2)-2\omega^2\right]}{(\omega+2)^2}-\sum_{i=1}^{n}\frac{\begin{matrix}2\left[(\omega+x_i)^2+(\varphi+1)(\varphi+2)\right]\\-4(\omega+x_i)^2\end{matrix}}{\left\{(\omega+x_i)^2+(\varphi+1)(\varphi+2)\right\}^2}+(\varphi+3)\sum_{i=1}^{n}\frac{1}{(\omega+x_i)^2}$$

The following equation can be solved for MLE's of $\hat{\varphi}$ and $\hat{\omega}$ of G-AD

$$\begin{pmatrix}\frac{\partial^2\ln L(\varphi,\omega)}{\partial\varphi^2} & \frac{\partial^2\ln L(\varphi,\omega)}{\partial\varphi\partial\omega}\\ \frac{\partial^2\ln L(\varphi,\omega)}{\partial\omega\partial\varphi} & \frac{\partial^2\ln L(\varphi,\omega)}{\partial\omega^2}\end{pmatrix}_{\substack{\hat{\varphi}=\varphi_0\\ \hat{\omega}=\omega_0}}\begin{pmatrix}\hat{\varphi}-\varphi_0\\ \hat{\omega}-\omega_0\end{pmatrix}=\begin{pmatrix}\frac{\partial\ln L(\varphi,\omega)}{\partial\varphi}\\ \frac{\partial\ln L(\varphi,\omega)}{\partial\omega}\end{pmatrix}_{\substack{\hat{\varphi}=\varphi_0\\ \hat{\omega}=\omega}}$$

where φ_0 and ω_0 are initial value of φ and ω respectively. The iterative method used to solve above non-linear equations is Newton-Raphson method available in R-software. The initial values of the parameters taken in this paper for estimating parameters are $\varphi_0=0.5$ and $\omega_0=0.5$.

3.3 Estimation of the stress-strength parameters

Following the approach given in Ray & Shanker (2023a), in this section our objective is to estimate $R=P\left(X>Y\right)$ when $X \sim$ G-AD$\left(\varphi_1,\omega_1\right)$ and $Y \sim$ G-AD$\left(\varphi_2,\omega_2\right)$, where X and Y are independently distributed strength and stress variables. Then, the Stress- Strength Parameter is given by

$$R=P\left(X>Y\right)=\int_0^{\infty}P\left(X>Y\mid Y=y\right)f_Y\left(y\right)dy$$

$$=\int_0^{\infty}\left[1-F_X\left(y\right)\right]f_Y\left(y\right)dy$$

$$=1-\int_0^{\infty}\frac{y^{\varphi_1}\left[\left[\left(\omega_1^{\,2}+2\right)y^2+2\left(\omega_1^{\,2}+\varphi_1+2\right)\omega_1 y+\left(\omega_1^{\,2}+\varphi_1^{\,2}+3\varphi_1+2\right)\omega_1^{\,2}\right]\right] \times\varphi_2\omega_2^{\,3}\left[\left(\omega_2+y\right)^2+\left(\varphi_2+1\right)\left(\varphi_2+2\right)\right]y^{\varphi_2-1}}{\left(\omega_1^{\,2}+2\right)\left(\omega_1+y\right)^{\varphi_1+2}\left(\omega_2^{\,2}+2\right)\left(\omega_2+y\right)^{\varphi_2+3}}dy$$

$$=1-\int_0^{\infty}\frac{\varphi_2\omega_2^{\,3}}{\left(\omega_1^{\,2}+2\right)\left(\omega_2^{\,2}+2\right)}$$

$$\times\frac{y^{\varphi_1+\varphi_2-1}\left[\left(\omega_1^{\,2}+2\right)y^2+2\left(\omega_1^{\,2}+\varphi_1+2\right)\omega_1 y+\left(\omega_1^{\,2}+\varphi_1^{\,2}+3\varphi_1+2\right)\omega_1^{\,2}\right]\times\left[\left(\omega_2+y\right)^2+\left(\varphi_2+1\right)\left(\varphi_2+2\right)\right]}{\left(\omega_1+y\right)^{\varphi_1+2}\left(\omega_2+y\right)^{\varphi_2+3}}dy$$

$$=\mathrm{H}\left(\varphi_1,\varphi_2,\omega_1,\omega_2\right)$$

Let, $\left(x_1,x_2,...,x_n\right)$ be the observed value of a random sample of size n from G-AD$\left(\varphi_1,\omega_1\right)$ and $\left(y_1,y_2,...,y_m\right)$ be the observed value of a random sample of size m from G-AD$\left(\varphi_2,\omega_2\right)$.

The log-likelihood functions of $\varphi_1,\varphi_2,\omega_1$ and ω_2 is given by

$$\ln L\left(\varphi_1,\varphi_2,\omega_1,\omega_2\right)$$

$$=n\ln\left(\varphi_1\right)+3n\ln\left(\omega_1\right)-n\ln\left(\omega_1^{\,2}+2\right)+\sum_{i-1}^{n}\ln\left[\left(\omega_1+x_i\right)^2+\left(\varphi_1+1\right)\left(\varphi_1+2\right)\right]$$

$$+\left(\varphi_1-1\right)\sum_{i=1}^{n}\ln\left(x_i\right)-\left(\varphi_1+3\right)\sum_{i=1}^{n}\ln\left(\omega_1+x_i\right)+m\ln\left(\varphi_2\right)+3m\ln\left(\omega_2\right)-m\ln\left(\omega_2^{\,2}+2\right)$$

$$+\sum_{i-1}^{m}\ln\left[\left(\omega_2+y_i\right)^2+\left(\varphi_2+1\right)\left(\varphi_2+2\right)\right]+\left(\varphi_2-1\right)\sum_{i=1}^{m}\ln\left(y_i\right)-\left(\varphi_2+3\right)\sum_{i=1}^{m}\ln\left(\omega_2+y_i\right)$$

Now,

$$\frac{\partial}{\partial\varphi_1}\left(\ln L\left(\varphi_1,\varphi_2,\omega_1,\omega_2\right)\right)=\frac{n}{\varphi_1}+\sum_{i=1}^{n}\frac{2\varphi_1+3}{\left(\omega_1+x_i\right)^2+\left(\varphi_1+1\right)\left(\varphi_1+2\right)}+\sum_{i=1}^{n}\ln\left(x_i\right)-\sum_{i=1}^{n}\ln\left(\omega_1+x_i\right)=0$$

$$\frac{\partial}{\partial\varphi_2}\left(\ln L\left(\varphi_1,\varphi_2,\omega_1,\omega_2\right)\right)=\frac{m}{\varphi_2}+\sum_{i=1}^{m}\frac{2\varphi_2+3}{\left(\omega_2+y_i\right)^2+\left(\varphi_2+1\right)\left(\varphi_2+2\right)}+\sum_{i=1}^{m}\ln\left(y_i\right)-\sum_{i=1}^{m}\ln\left(\omega_2+y_i\right)=0$$

$$\frac{\partial}{\partial\omega_1}\left(\ln L\left(\varphi_1,\varphi_2,\omega_1,\omega_2\right)\right)=\frac{3n}{\omega_1}-\frac{2\omega_1 n}{\left(\omega_1^{\,2}+2\right)}+\sum_{i=1}^{n}\frac{2\left(\omega_1+x_i\right)}{\left(\omega_1+x_i\right)^2+\left(\varphi_1+1\right)\left(\varphi_1+2\right)}-\left(\varphi_1+3\right)\sum_{i=1}^{n}\frac{1}{\left(\omega_1+x_i\right)}=0$$

$$\frac{\partial}{\partial \omega_2}\left(\ln L\left(\varphi_1,\varphi_2,\omega_1,\omega_2\right)\right)=\frac{3m}{\omega_2}-\frac{2\omega_2 m}{\left(\omega_2^{\,2}+2\right)}+\sum_{i=1}^{m}\frac{2\left(\omega_2+y_i\right)}{\left(\omega_2+y_i\right)^2+\left(\varphi_2+1\right)\left(\varphi_2+2\right)}-\left(\varphi_2+3\right)\sum_{i=1}^{m}\frac{1}{\left(\omega_2+y_i\right)}=0$$

Solving these non-linear equations using any iterative methods available in R packages , we can obtain the MLEs of the parameters as $\left(\hat{\varphi}_1,\hat{\varphi}_2,\hat{\omega}_1,\hat{\omega}_2\right)$ and hence the MLE of R can thus be obtained as

$$\hat{R}=\mathrm{H}\left(\hat{\varphi}_1,\hat{\varphi}_2,\hat{\omega}_1,\hat{\omega}_2\right)$$

3.4 A Simulation Study

This section contains a simulation study to examine the consistency of maximum likelihood estimators of the G-AD. Following the simulation procedure based on acceptance rejection method given in Shanker *et al.* (2023) has been used to compute the mean, bias (B), MSE and variance of the MLE's.

Table 1. The mean, Biases, MSE and Variances of G-AD for $\varphi=5.5$, $\omega=0.6$

Parameters	Sample Size	Mean	Bias	MSE	Variance
$\hat{\varphi}$	25	5.777858	0.277858	0.078268	0.00106374
	50	5.773558	0.273557	0.076207	0.00137373
	100	5.771150	0.271150	0.076071	0.00254874
	150	5.768298	0.268298	0.073654	0.00167103
	200	5.767962	0.267962	0.073415	0.00161190
$\hat{\omega}$	25	0.575987	-0.024012	0.000583	0.00007209
	50	0.576822	-0.023177	0.000552	0.00001496
	100	0.576913	-0.023086	0.000549	0.00001648
	150	0.577096	-0.022903	0.000544	0.00001946
	200	0.577178	-0.022821	0.000536	0.00001604

Tables 1 and 2 reveal that for increasing sample size, the value of the biases, MSE and variances of the MLE of the parameters of G-AD becoming smaller and certify the first-order asymptotic theory of maximum likelihood estimators.

Table 2. The mean, Biases, MSE and Variances of G-AD for $\varphi=12.5$, $\omega=5.4$

Parameters	Sample Size	Mean	Bias	MSE	Variance
$\hat{\varphi}$	25	12.35329	-0.1467112	0.027497	0.005973
	50	12.35961	-0.1403934	0.025734	0.006020
	100	12.36633	-0.1336662	0.023351	0.005485
	150	12.37669	-0.1233076	0.022522	0.007317
	200	12.38015	-0.1198541	0.020441	0.007317
$\hat{\omega}$	25	5.290178	-0.1098215	0.013987	0.002371
	50	5.290988	-0.1090125	0.013549	0.001666
	100	5.292633	-0.1073667	0.012883	0.001355
	150	5.298923	-0.1010770	0.011377	0.001161
	200	5.300818	-0.0991824	0.010819	0.000982

4. Results and Discussion

This section deals with the goodness of fit of G-AD over G-LD, G-SD, Weibull and gamma distributions to illustrate its applications and using two real datasets relating to survival time of patients of acute bone cancer and head and neck cancer. The summary of the two datasets is presented in tables 3 and 4 respectively. The total time to test (TTT) plots of the two datasets are given in figures 5 and 6 respectively. The goodness of fit of the considered distributions for two datasets is provided in tables 5 and 6 respectively. The fitted plots of the considered distributions for the two datasets are given in figure 7. The p-p plots of the considered distributions for the two datasets are finally presented in figures 8 and 9 respectively. The datasets are as follows:

Dataset 1: Acute bone cancer: This dataset represents the survival times (in days) of 73 patients Who diagnosed with acute bone cancer available in Mansour et al (2020) and are as follows:
0.09, 0.76, 1.81, 1.10, 3.72, 0.72, 2.49, 1.00, 0.53,0.66, 31.61, 0.60, 0.20, 1.61, 1.88, 0.70, 1.36, 0.43,3.16, 1.57, 4.93, 11.07, 1.63, 1.39, 4.54, 3.12,86.01, 1.92, 0.92, 4.04, 1.16, 2.26, 0.20, 0.94, 1.82, 3.99,1.46, 2.75, 1.38, 2.76, 1.86, 2.68, 1.76,0.67, 1.29, 1.56, 2.83, 0.71, 1.48, 2.41, 0.66, 0.65, 2.36, 1.29,13.75, 0.67, 3.70, 0.76, 3.63, 0.68,2.65, 0.95, 2.30, 2.57, 0.61, 3.93, 1.56, 1.29, 9.94, 1.67, 1.42, 4.18,1.37.

Table 3. The summary of acute bone cancer dataset

Min	1st Qu.	Median	Mean	Variance	3rd Qu.	Max
0.090	0.920	1.570	3.755	112.33	2.750	86.010

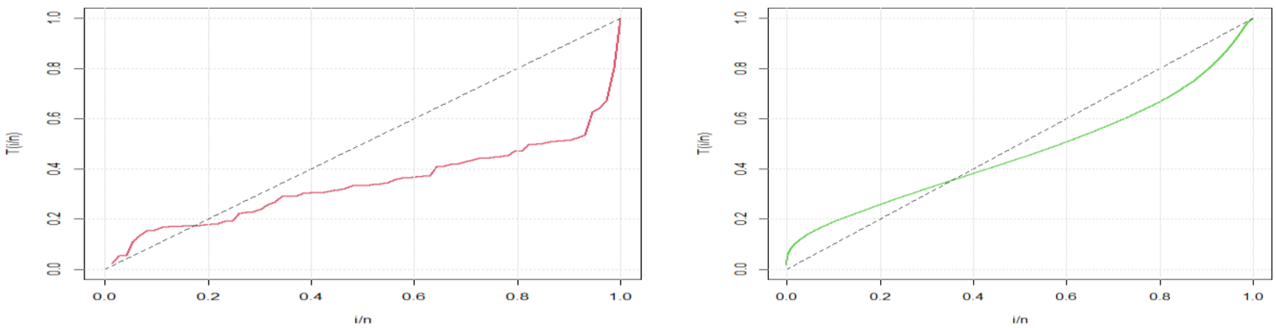

Figure 5. TTT-plot of the acute bone cancer dataset and simulated data of G-AD.

Dataset 2: Head and Neck cancer: This dataset is the survival time of 44 patients diagnosed by Head and Neck cancer disease are available in Efron (1988) and are given by
12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194,195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776

Table 4. The summary of head and neck cancer dataset

Min	1st Qu.	Median	Mean	Variance	3rd Qu.	Max
12.20	67.21	128.50	223.48	93286.41	219.00	1776.00

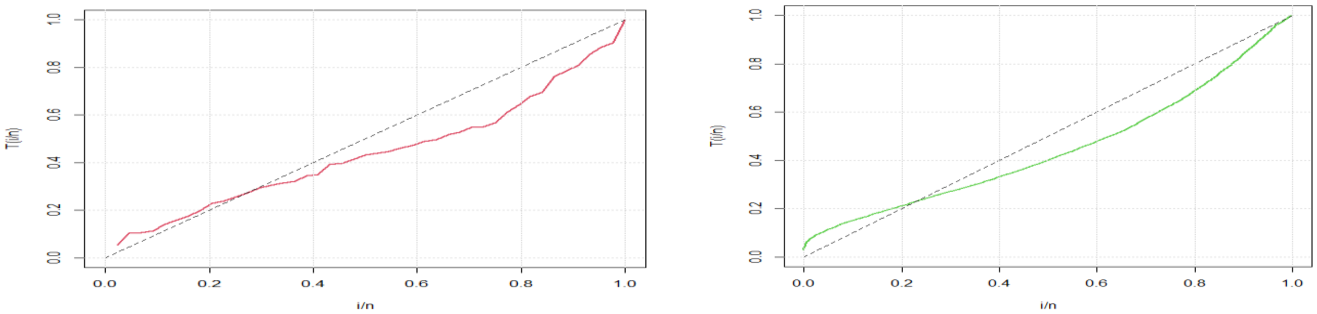

Figure 6. TTT-plot of the head and neck cancer dataset and simulated data of G-AD.

Table 5. Goodness of fit of probability models for acute bone cancer dataset

Distributions	ML estimates $\hat{\varphi}(S.E\,of\,\hat{\varphi})$ $\hat{\omega}(S.E\,of\,\hat{\omega})$	$-2\log L$	AIC	BIC	K-S	p- value
G-AD	4.8263(1.0872) 0.7430(0.1502)	282.3014	286.3014	300.6114	0.08	0.48
G-SD	4.8969(1.3904) 0.4967(0.1360)	282.8051	286.8051	301.1151	0.10	0.39
G-LD	5.1600(1.8468) 0.4375(0.1602)	284.315	288.315	302.625	0.11	0.33
Gamma	0.1985(0.0389) 0.7456(0.1057)	334.5311	338.5311	352.8411	0.56	0.00
Weibull	0.4395(0.0687) 0.7655(0.0567)	322.8033	326.8033	341.1133	0.25	0.00

Table 6. Goodness of fit of probability models for head and neck cancer dataset

Distributions	ML estimates $\hat{\varphi}(S.E\,of\,\hat{\varphi})$ $\hat{\omega}(S.E\,of\,\hat{\omega})$	$-2\log L$	AIC	BIC	K-S	p- value
G-AD	9.6688(17.4771) 9.2503(16.9773)	558.4853	562.4853	576.7953	0.07	0.95
G-SD	8.6787(11.7435) 10.0923(14.851)	558.4641	562.4641	576.7741	0.09	0.81
G-LD	8.4483(10.4902) 11.1557(14.3688)	558.4555	562.4555	576.7655	0.09	0.7
Gamma	0.0047(0.0010) 1.0522(0.1886)	564.0254	568.0254	582.3354	1.00	0.00
Weibull	0.0070(0.0034) 0.9234(0.0809)	563.7155	567.7155	582.0255	0.5	0.04

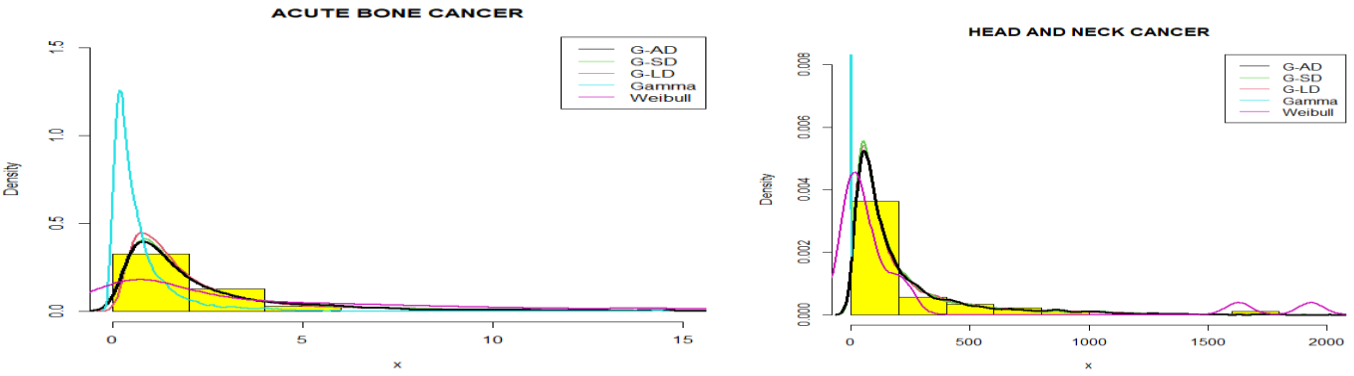

Figure 7. Fitted plots of distributions for acute bone cancer and head and neck cancer datasets.

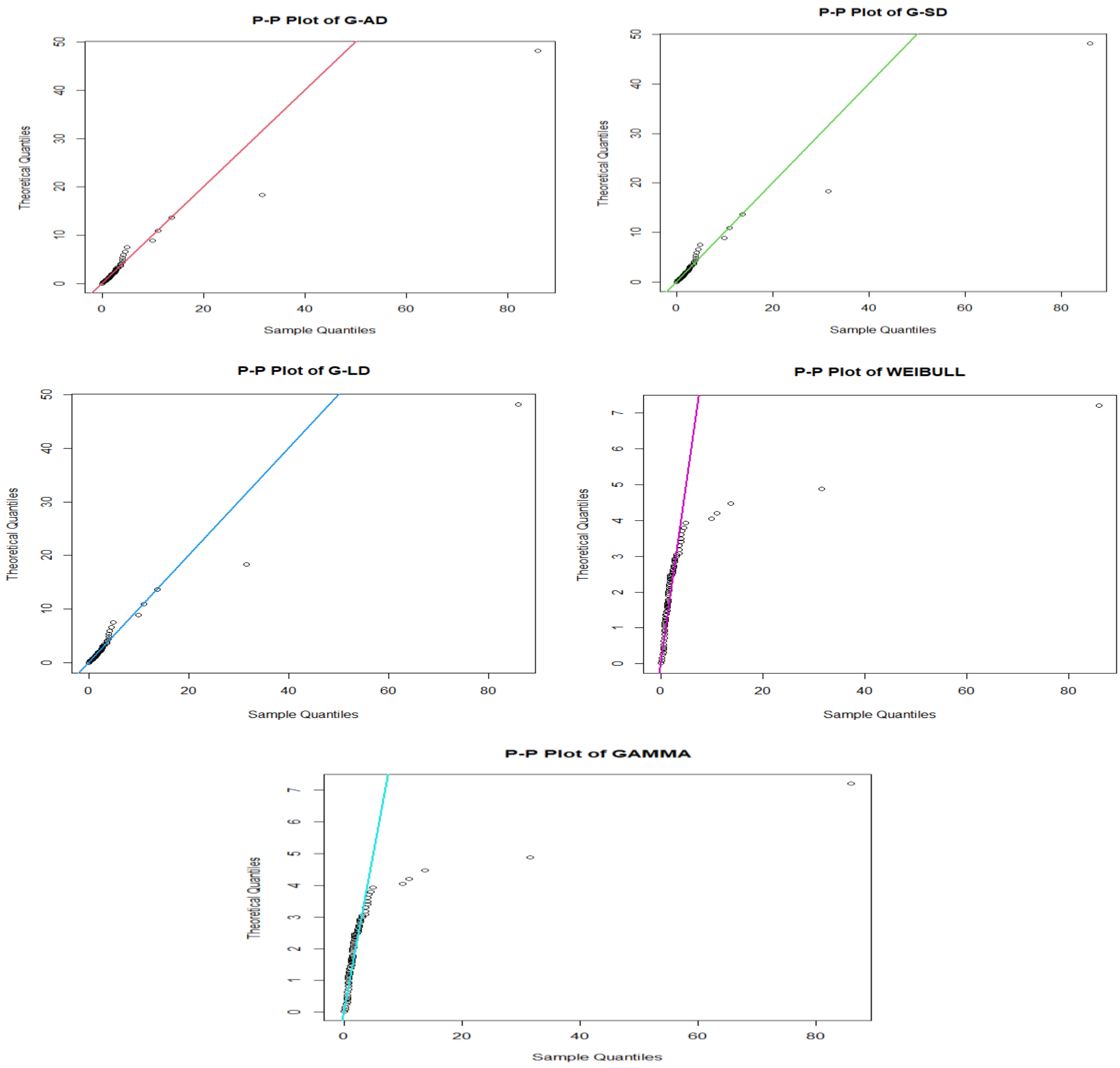

Figure 8. P-P plots for considerd distributions of acute bone cancer dataset.

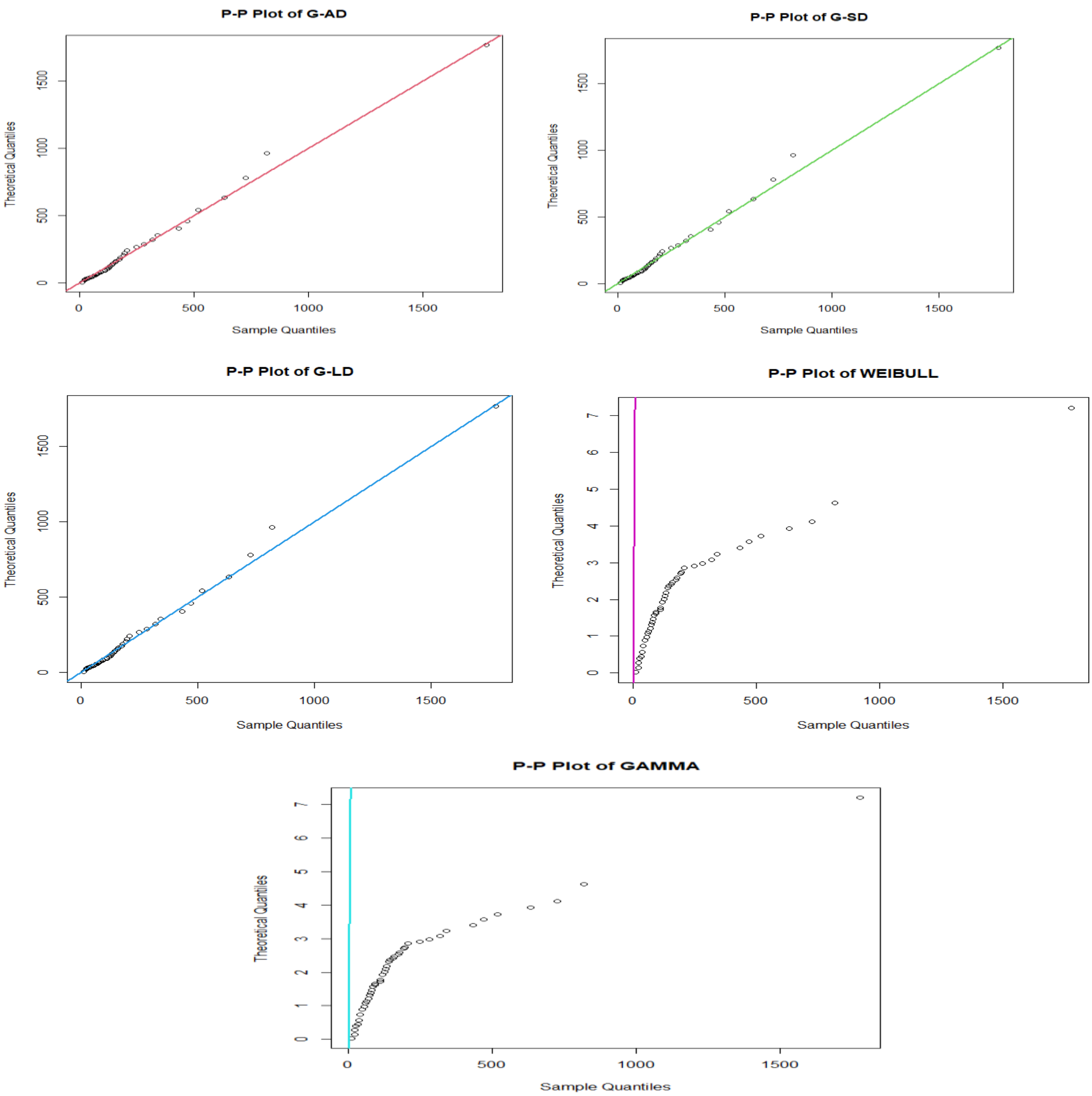

Figure 9. P-P plots for considerd distributions of head and neck cancer dataset.

From the summary of the two datasets in tables 3 and 4, it is quite obvious that the considered datasets are highly positively skewed and highly over-dispersed. Based on the values of $-2\log L$, AIC (Akaike information criterion), Kolmogorov – Smirnov (K-S) statistic and the fitted plots of two-parameter lifetime distributions, it is crystal clear from the goodness of fit that two-parameter G-AD is the best for modeling survival times of patients suffering from acute bone cancer and head and neck cancer. It can be recalled that recently Klakattawi (2022) proposed a new extended Weibull distribution with five parameters and used it for analyzing survival times of cancer patients and found that it provides much better fit than several two-parameter, three-parameter ,four-parameter and five-parameter lifetime distributions including Weibull distribution, alpha power Weibull (APW) distribution by Nassar *et al.* (2017), Beta-Weibull (BW) distribution by Famoye *et al.* (2005),Kumaraswamy-Weibull (Kum-W) distribution by Cordeiro *et al.* (2010), exponentiated generalized Weibull (EGW) distribution by Cordeiro *et al.* (2013), a new

Kumaraswamy family of generalized Weibull distribution by Ahmed *et al.* (2015) and exponentiated Kumaraswamy Weibull distribution by Eissa (2017), some among others. Here we would like to emphasize that the proposed gamma-Akash distribution (G-AD) provides much closure fit than all these two-parameter, three-parameter, four-parameter and five-parameter lifetime distributions as it can be seen from the test of goodness of fit given by Klakattawi (2022). The most interesting feature of G-AD is that being two-parameter distribution is much easier to characterize and handle the distribution as compared to three-parameter, four-parameter and five parameter distributions and hence it can be considered an important probability model for modeling survival time of cancer patients.

5. Conclusions

In this paper, we propose a gamma-Akash probability model, a compound of gamma and Akash distribution to model data of long tails. Some important statistical and reliability properties have been discussed. Maximum likelihood estimation has been discussed for estimating parameters and simulation studies to know the consistency of ML estimators are presented. The goodness of fit of the G-AD has been compared with several well-known two-parameter distributions and observed that the fit was better than that obtained for gamma and Weibull distributions and slightly superior to those obtained for G-LD and G-SD and hence it can be considered as an important probability models for survival time of patients suffering from acute bone cancer and head and neck cancer in biomedical science. As the proposed distribution is the new probability model, a lot of works can be done in the future and definitely it will draw the attention of research workers in biomedical sciences and biomedical engineering.

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Author Contributions

Conceptualization: SHANKER, R.; **Data curation:** MOUSUMI, R.; **Formal analysis:** MOUSUMI, R.; **Funding acquisition:** - **Investigation:** SHANKER, R.; **Methodology:** SHANKER, R.; MOUSUMI, R.; **Project administration:** SHANKER, R.; **Software:** - **Resources:** MOUSUMI, R.; **Supervision:** SHANKER, R.; **Validation:** SHANKER, R.; **Visualization:** MOUSUMI, R.; **Writing - original draft:** SHANKER, R.; MOUSUMI, R.; **Writing - review and editing:** SHANKER, R.; MOUSUMI, R.;

Conflicts of Interest

There is no conflict of interest.

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