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ARTICLE

A novel ratio cum product type exponential class of estimators of finite population mean in Adaptive cluster Sampling[1]

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Abstract
In the present paper a ratio cum product type exponential class of estimators has been proposed to estimate the finite population mean of rare type or hard to reach type population. The mean square error and bias expressions of the proposed generalized class have been derived and presented up to the first order of approximation. New estimators have been developed from the proposed class using robust measures. Using simulation study and a real data application, the efficiency of the newly developed estimators from the class that is proposed have been shown. The results show that the new developed estimators are more efficient than the competing estimators presented in this paper.

Keywords: Combining estimators; Ratio-cum-product; Regression estimator; Hidden clustered event; Rare event.

1. Introduction

There are various sampling designs to be used depending on the population that is being studied namely SRSWOR (Simple random sampling without replacement), StRS (Stratified random sampling), Cluster sampling, Systematic sampling among others. In some situations, a combination of these designs is also used which are termed as complex sampling designs to acquire a sample that is representative of the population. However, when the study population is rare or hidden clustered, obtaining a representative sample becomes a problem and in such a case, the Adaptive Cluster Sampling Design abbreviated as ACS design is used. The ACS design was proposed by Thompson (1990) which allows the researchers to obtain a representative sample by using a pre-defined condition to select the rare event's observations.

Since the methodology of the ACS (Thompson, 1990) design was complex, not much work in developing efficient estimators was done prior to 2007 compared to the classical sampling designs like Simple random sampling without replacement design (SRSWOR), Stratified random sampling among others. Motivated from the work of Cochran (1940), many researchers started developing estimators using auxiliary information which resulted in the development of fundamental estimators namely ratio (Cochran, 1940), product (Murthy, 1964) and regression (Hansen *et al.*, 1953) type estimators. Apart from these fundamental estimators, we point out here some notable research works which introduced new ideas in developing more efficient estimators (Bahl & Tuteja, 1991; Khoshnevisan *et al.*, 2007; Grover & Kaur, 2011; Singh *et al.*, 2016; Ahmad *et al.*, 2021; Ahmad *et al.*, 2022; Ahmad *et al.*, 2023). As pointed above, due to the complexity of the ACS design not much work was done in developing efficient estimators but the proposal of transformed population approach by Dryver & Chao (2007) brought ACS back to the Simple random sampling without replacement and since then research for developing efficient estimators started to pick up the pace. The transformed population approach proposed by Dryver & Chao (2007) allows the use of network means of networks observed as per SRSWOR for estimation and thus using the transformed population, we can use the standard results of SRSWOR design for ACS.

We recommend the readers (Dryver & Chao, 2007) for a detailed discussion on the transformed population approach. After the research of Dryver & Chao (2007), many ratio type estimators have been proposed in the ACS design. Dryver & Chao (2007) proposed the classical ratio estimator utilizing one auxiliary variable. Chutiman (2013) developed some transformed ratio type estimators using one auxiliary variable and some of its parameters considered known. Yadav *et al.* (2016) developed several new improved estimators for estimating population mean using one auxiliary variable and some of its parameters considered as known namely the coefficients of skewness, the coefficients of kurtosis and the coefficient of correlation among the auxiliary and survey variable. Singh *et al.* (2024) proposed multiple log type estimators utilizing one auxiliary variable and different known values of parameters of that auxiliary variable. Raghav *et al.* (2024) developed several efficient classes for mean and variance estimation and studied their properties. Mishra *et al.* (2024) developed a class by combining ratio and product forms. Qureshi *et al.* (2018) developed a ratio type class of estimator utilizing one auxiliary variable along with several of its known values of parameters and robust measures. Singh & Mishra (2023) developed some novel estimators using two auxiliary variables in ACS. Singh & Mishra (2022) proposed some improved exponential type estimator in ACS and analyzed its performance. In a recent development, Raghav *et al.* (2023) study the two-phase ACS design under the transformed population approach and developed some new estimators.

In survey sampling, the choice of estimators is based on the relationship between the survey and auxiliary variable. But, Singh & Espejo (2003) proposed an estimator by using the ratio and the product form together which allows use of the estimator proposed by Singh & Espejo (2003) for both the cases when the correlation between auxiliary and survey variable is strong and positive as well as when the correlation between auxiliary and survey variable is strong and negative. Coping with the idea, Singh *et al.* (2016) created a RCP (ratio cum product) type class of estimators in SRSWOR design using the exponential method. Dansawad (2023) developed a chain type exponential estimator coping with the idea of Singh *et al.* (2016). Such type of estimators has been comparatively unexplored in the ACS design and therefore in this paper, motivated from the RCP type estimators of Singh & Espejo (2003), Singh *et al.* (2016) and the work of Dansawad (2023) we have developed a novel RCP type exponential class of estimators incorporating a exponential function in the traditional RCP structure in ACS design and studied its properties.

The rest of the paper is presented as: The Section 2 comprises of the proposed RCP (ratio cum product) type exponential class estimators together with the derivation of its mean squared error and bias up to first order approximations. In this same section, the new ratio-cum-product type exponential estimators which have been developed from the proposed novel RCP type exponential class of estimators have been provided. In Section 3, a study on mathematical comparison of MSEs of the existing estimators and the proposed class of estimators has been provided. In Section 4, in order to study and compare the efficiency of our newly created estimators against related competing existing estimators which are provided in Table 2, a simulation study is conducted and the results of the simulation study show that the new developed estimators perform better. In section 5, the novelty of the new developed estimators is shown using a real-life example where the new developed estimators are used to estimate the average number of thorny plants of the plateaus belonging to the Western Ghats of Sahyadri from Goa to Varandha Ghat (Bhor, Maharashtra, India) (see Latpate & Kshirsagar, 2020) using percentage of aluminum as auxiliary variable. In Section 6 results and discussion on this paper are provided.

2. Proposed Novel Ratio-cum-Product Exponential Class

Following the work of Singh & Espejo (2003), Singh *et al.* (2016) and Dansawad (2023) we now propose the class as follows:

$$t_G = \bar{w}_y \left(\frac{a\mu x + b}{a\bar{w}_x + b}\right)^{(2g-1)} exp\left(\frac{a(\mu_x - \bar{w}_x)}{a(\mu_x + \bar{w}_x) + 2b}\right), \quad (1)$$

In this context, a and b are constants chosen with precision to ensure that a range of both new and existing related estimators can be incorporated into the proposed class. The selection of these constants is critical for including various estimators that might otherwise fall outside the class. Additionally, g is optimized to achieve the lowest possible mean square error (MSE) for the proposed class t_G. This optimization process aims to ensure that the estimators in this class perform as accurately as possible.

It is also essential to understand that a and b are assigned specific values based on known parameters of the auxiliary variable. These parameters can include values such as the coefficient of skewness, kurtosis, and other relevant statistics. By adjusting a and b according to these known parameters, we can tailor the estimators within the proposed class to effectively utilize the information provided by the auxiliary variable, thereby enhancing their overall performance and accuracy.

Using the error terms we re-write equation (1) as:

$$t_G = \mu_y(e_{w_y} + 1)(1 + \tau e_{w_x})^{(1-2g)} exp\left(\frac{\tau}{2} e_{w_x}\left(1 + \frac{1}{2}\tau e_{w_x}\right)\right), \quad (2)$$

where $\tau = \frac{a\mu_x}{a\mu_x + b}$.

On simplifying, we get:

$$t_G - \mu_y \quad = \mu_y \begin{pmatrix} e_{w_y} + \left(\frac{1}{2} - 2g\right)\tau e_{w_x} + \left(-\frac{1}{8} + 2g^2\right)\tau^2 e_{w_x}^2 \\ + \left(\frac{1}{2} - 2g\right)\tau e_{w_y} e_{w_x} \end{pmatrix} \tag{3}$$

Upon using the expectation function on both sides, we get:

$$Bias(t_G) = \mu_y \left(\left(-\frac{1}{8} + 2g^2\right)\tau^2 f C_{w_x}^2 + \left(\frac{1}{2} - 2g\right)\tau f \rho_{w_y w_x} C_{w_y} C_{w_x}\right). \tag{4}$$

Upon raising to the power of two and using the expectation function in (3), we get:

$$MSE(t_G) = \mu_y^2 \begin{pmatrix} f C_{w_y}^2 + \left(\frac{1}{4} + 4g^2 - 2g\right)\tau^2 f C_{w_x}^2 \\ +2\left(\frac{1}{2} - 2g\right)\tau f \rho_{w_y w_x} C_{w_y} C_{w_x} \end{pmatrix} \tag{5}$$

We partially differentiating (5) and equate the resultant to zero and we get:

$$g_{opt} = \frac{1}{4}\left[\frac{2}{\tau}\rho_{w_y w_x}\frac{C_{w_y}}{C_{w_x}} + 1\right]. \tag{6}$$

Using (6) in (5) we get:

$$MSE(t_{G_{min}}) = f S_{w_y}^2 (1 - \rho_{w_y w_x}^2). \tag{7}$$

It is noteworthy that the minimum mean square error (MSE) of our proposed class of the novel RCP type exponential estimator, denoted as $MSE(t_{G_{min}})$, is identical to the regression type estimator's minimum MSE developed by Chutiman (2013). This indicates that the estimators we have proposed can be used as an alternative to Chutiman's regression estimator when evaluating performance based on the MSE criterion. This equivalence suggests that our estimators are competitive with, and potentially as effective as, Chutiman's regression estimator in terms of minimizing MSE.

According to equation (6), achieving the minimum MSE requires knowledge of the parameters C_{w_y}, C_{w_x} and $\rho_{w_y w_x}$. These parameters can either be determined through additional costs or obtained from prior surveys if they are not readily available. The necessity to accurately know these parameters emphasizes the importance of their estimation in practical applications.

The primary objective of this paper was to explore, develop and study novel RCP type exponential estimators in the context of ACS (Adaptive Cluster Sampling) design. To fulfill this objective, we have constructed a variety of RCP type exponential estimators from the proposed class, incorporating robust measures such as the mid-range and Hodges-Lehmann estimator. Additionally, we utilized different known parameters of the auxiliary variable, including the coefficient of skewness and the coefficient of kurtosis.

The resulting new estimators, derived from these methodologies, are detailed in the table below. These estimators have been developed to offer alternative approaches for estimation within the ACS design framework and to provide options that can potentially enhance estimation accuracy and reliability.

Table 1. New developed estimators from class t_G

t_*	a	b	τ
$t_{G_1} = \bar{w}_y \left(\frac{MR\mu x+\beta_1(w_x)}{MR\bar{w}_x+\beta_1(w_x)}\right)^{(2g_1-1)} exp\left(\frac{MR(\mu_x-\bar{w}_x)}{MR(\mu_x+\bar{w}_x)+2\beta_1(w_x)}\right)$	MR	$\beta_1(w_x)$	$\frac{MR\mu_x}{MR\mu_x+\beta_1(w_x)}$
$t_{G_2} = \bar{w}_y \left(\frac{HL\mu x+\beta_1(w_x)}{HL\bar{w}_x+\beta_1(w_x)}\right)^{(2g_2-1)} exp\left(\frac{HL(\mu_x-\bar{w}_x)}{HL(\mu_x+\bar{w}_x)+2\beta_1(w_x)}\right)$	HL	$\beta_1(w_x)$	$\frac{HL\mu_x}{HL\mu_x+\beta_1(w_x)}$
$t_{G_3} = \bar{w}_y \left(\frac{HL\mu x+\beta_2(w_x)}{HL\bar{w}_x+\beta_2(w_x)}\right)^{(2g_3-1)} exp\left(\frac{HL(\mu_x-\bar{w}_x)}{HL(\mu_x+\bar{w}_x)+2\beta_2(w_x)}\right)$	HL	$\beta_2(w_x)$	$\frac{HL\mu_x}{HL\mu_x+\beta_2(w_x)}$
$t_{G_4} = \bar{w}_y \left(\frac{MR\mu x+\beta_2(w_x)}{HL\bar{w}_x+\beta_2(w_x)}\right)^{(2g_4-1)} exp\left(\frac{MR(\mu_x-\bar{w}_x)}{MR(\mu_x+\bar{w}_x)+2\beta_2(w_x)}\right)$	MR	$\beta_2(w_x)$	$\frac{MR\mu_x}{MR\mu_x+\beta_2(w_x)}$

Table 2. Competing estimators and their Mean squared errors

Existing estimators	MSE
$t_{Th} = \bar{w}_y$(Thompson, 1990)	$fS^2_{w_y}$
$t_{DC} = \frac{\bar{w}_y}{\bar{w}_x}\mu_x$ (Dryver and Chao, 2007)	$f\mu^2_Y(C^2_{w_y} + C^2_{w_x} - 2\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{CH_1} = \bar{w}_y\left(\frac{\mu_x+C_{w_x}}{\bar{w}_x+C_{w_x}}\right)$ (Chutiman, 2013)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{\mu_x}{\mu_x+C_{w_x}}\right)^2 C^2_{w_x} - 2\left(\frac{\mu_x}{\mu_x+C_{w_x}}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{CH_2} = \bar{w}_y\left(\frac{\beta_2(w_x)\mu_x+C_{w_x}}{\beta_2(w_x)\bar{w}_x+C_{w_x}}\right)$ (Chutiman, 2013)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{\mu_x\beta_2(w_x)}{\mu_x\beta_2(w_x)+C_{w_x}}\right)^2 C^2_{w_x} - 2\left(\frac{\mu_x\beta_2(w_x)}{\mu_x\beta_2(w_x)+C_{w_x}}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{CH_3} = \bar{w}_y\left(\frac{\mu_x+\beta_2(w_x)}{\bar{w}_x+\beta_2(w_x)}\right)$ (Chutiman, 2013)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{\mu_x}{\mu_x+\beta_2(w_x)}\right)^2 C^2_{w_x} - 2\left(\frac{\mu_x}{\mu_x+\beta_2(w_x)}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{YS_1} = \bar{w}_y\left(\frac{\beta_2(w_x)\mu_x+\beta_1(w_x)}{\beta_2(w_x)\bar{w}_x+\beta_1(w_x)}\right)$ (Yadav *et al.*, 2016)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{\beta_2(w_x)\mu_x}{\beta_2(w_x)\mu_x+\beta_1(w_x)}\right)^2 C^2_{w_x} - 2\left(\frac{\beta_2(w_x)\mu_x}{\beta_2(w_x)\mu_x+\beta_1(w_x)}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{YS_2} = \bar{w}_y\left(\frac{\beta_1(w_x)\mu_x+\beta_2(w_x)}{\beta_1(w_x)\bar{w}_x+\beta_2(w_x)}\right)$ (Yadav *et al.*, 2016)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{\beta_1(w_x)\mu_x}{\beta_1(w_x)\mu_x+\beta_2(w_x)}\right)^2 C^2_{w_x} - 2\left(\frac{\beta_1(w_x)\mu_x}{\beta_1(w_x)\mu_x+\beta_2(w_x)}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{QK_1} = \bar{w}_y\left(\frac{MR\mu_x+\beta_1(w_x)}{MR\bar{w}_x+\beta_1(w_x)}\right)$ (Qureshi *et al.*, 2018)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{MR\mu_x}{MR\mu_x+\beta_1(w_x)}\right)^2 C^2_{w_x} - 2\left(\frac{MR\mu_x}{MR\mu_x+\beta_1(w_x)}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{QK_2} = \bar{w}_y\left(\frac{MR\mu_x+TM}{MR\bar{w}_x+TM}\right)$ (Qureshi *et al.*, 2018)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{MR\mu_x}{MR\mu_x+TM}\right)^2 C^2_{w_x} - 2\left(\frac{MR\mu_x}{MR\mu_x+TM}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{QK_3} = \bar{w}_y\left(\frac{HL\mu_x+\beta_1(w_x)}{HL\bar{w}_x+\beta_1(w_x)}\right)$ (Qureshi *et al.*, 2018)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{HL\mu_x}{HL\mu_x+\beta_1(w_x)}\right)^2 C^2_{w_x} - 2\left(\frac{HL\mu_x}{HL\mu_x+\beta_1(w_x)}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$
$t_{QK_4} = \bar{w}_y\left(\frac{HL\mu_x+TM}{HL\bar{w}_x+TM}\right)$ (Qureshi *et al.*, 2018)	$f\mu^2_Y(C^2_{w_y} + \left(\frac{HL\mu_x}{HL\mu_x+TM}\right)^2 C^2_{w_x} - 2\left(\frac{HL\mu_x}{HL\mu_x+TM}\right)\rho_{w_xw_y}C_{w_y}C_{w_x})$

3. Mathematical Mean Square Error Comparison

In this Section, mathematical comparison of MSEs of the existing estimators presented in Table-2 and the proposed class of estimators has been provided.

3.1 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{DC})$:

$$MSE(t_{G_{min}}) < MSE(t_{DC})$$
$$fS^2_{w_y}(1-\rho^2_{w_yw_x}) < f\mu^2_Y(C^2_{w_y} + C^2_{w_x} - 2\rho_{w_xw_y}C_{w_y}C_{w_x})$$

$$f\mu_Y^2 C_{w_y}^2(1-\rho_{w_y w_x}^2) < f\mu_Y^2(C_{w_y}^2 + C_{w_x}^2 - 2\rho_{w_x w_y} C_{w_y} C_{w_x})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + C_{w_x}^2 - 2\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

3.2 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{CH_1})$:

$$MSE(t_{G_{min}}) < MSE(t_{CH_1})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{CH_1}^2 C_{w_x}^2 - 2\theta_{CH_1}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\theta_{CH_1} = \frac{\mu_x}{\mu_x + C_{w_x}}$.

3.3 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{CH_2})$:

$$MSE(t_{G_{min}}) < MSE(t_{CH_2})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{CH_2}^2 C_{w_x}^2 - 2\theta_{CH_2}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\theta_{CH_2} = \frac{\mu_x \beta_2(w_x)}{\mu_x \beta_2(w_x) + C_{w_x}}$.

3.4 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{CH_3})$:

$$MSE(t_{G_{min}}) < MSE(t_{CH_3})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{CH_3}^2 C_{w_x}^2 - 2\theta_{CH_3}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\theta_{CH_3} = \frac{\mu_x}{\mu_x + \beta_2(w_x)}$.

3.5 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{YS_1})$:

$$MSE(t_{G_{min}}) < MSE(t_{YS_1})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{YS_1}^2 C_{w_x}^2 - 2\theta_{YS_1}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\theta_{YS_1} = \frac{\beta_2(w_x)\mu_x}{\beta_2(w_x)\mu_x + \beta_1(w_x)}$.

3.6 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{YS_2})$:

$$MSE(t_{G_{min}}) < MSE(t_{YS_2})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{YS_2}^2 C_{w_x}^2 - 2\theta_{YS_2}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\theta_{YS_2} = \frac{\beta_1(w_x)\mu_x}{\beta_1(w_x)\mu_x + \beta_2(w_x)}$.

3.7 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{QK_1})$:

$$MSE(t_{G_{min}}) < MSE(t_{QK_1})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{QK_1}^2 C_{w_x}^2 - 2\theta_{QK_1}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\theta_{QK_1} = \frac{MR\mu_x}{MR\mu_x + \beta_1(w_x)}$.

3.8 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{QK_2})$:

$$MSE(t_{G_{min}}) < MSE(t_{QK_2})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{QK_2}^2 C_{w_x}^2 - 2\theta_{QK_2}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\theta_{QK_2} = \frac{MR\mu_x}{MR\mu_x + TM}$.

3.9 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{QK_3})$:

$$MSE(t_{G_{min}}) < MSE(t_{QK_3})$$
$$C_{w_y}^2\rho_{w_y w_x}^2 + \theta_{QK_3}^2 C_{w_x}^2 - 2\theta_{QK_3}\rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\Theta_{QK_3} = \frac{HL\mu_x}{HL\mu_x + \beta_1(w_x)}$.

3.10 Comparison of $MSE(t_{G_{min}})$ and $MSE(t_{QK_4})$:

$$MSE(t_{G_{min}}) < MSE(t_{QK_4})$$

$$C_{w_y}^2 \rho_{w_y w_x}^2 + \Theta_{QK_4}^2 C_{w_x}^2 - 2\Theta_{QK_4} \rho_{w_x w_y} C_{w_y} C_{w_x} > 0$$

where $\Theta_{QK_4} = \frac{HL\mu_x}{HL\mu_x + TM}$.

4. Simulation Study

To study the efficiency of the developed ratio cum product type exponential estimators $t_{G_1} - t_{G_4}$ with respect to the related existing estimators discussed in this paper (see Table-2), we have conducted a simulation study. To compare the performance of the estimators, percentage relative efficiency (PRE) is used. The study is conducted in R software using the following algorithm:

1. Using the model $y_i = \frac{x_i}{4} + e_i$ where $e_i \sim N(0, x_i)$ the study population is generated. Observations for *X* which is the auxiliary variable are from (Thompson, 2012).
2. Samples of sizes *n* =130, 135, 140, 144 are obtained using the ACS procedure and several values of estimators are obtained.
3. Various *MSE* values are obtained for each sample size *n* using the formula $MSE(t_*) = \frac{1}{20000}\sum_{i=1}^{20000}\left(t_* - \mu_y^2\right)^2$ where t_* is the appropriate estimator.
4. Using the *MSE* values obtained, the *PRE* is calculated and the results are presented in Table-3.

Table 3. Percentage Relative Efficiencies obtained of the estimators

Estimators	*n*=130	*n*=135	*n*=140	*n*=144
t_{TH}	100.00	100.00	100.00	100.00
t_{DC}	214.01	215.34	219.16	220.48
t_{CH_1}	123.73	123.70	123.46	124.48
t_{CH_2}	186.78	185.82	186.65	188.43
t_{CH_3}	110.84	111.07	110.80	111.69
t_{YS_1}	188.90	187.94	188.85	190.65
t_{YS_2}	129.69	129.54	129.33	130.43
t_{QK_1}	180.04	179.11	179.71	181.45
t_{QK_2}	214.01	215.34	219.16	220.48
t_{QK_3}	192.61	191.67	192.74	194.55
t_{QK_4}	214.01	215.34	219.16	220.48
t_{G_1}	216.79	217.53	221.70	222.84
t_{G_2}	215.95	216.96	221.31	222.34
t_{G_3}	217.44	217.92	221.88	223.17
t_{G_4}	217.66	217.99	221.79	223.21

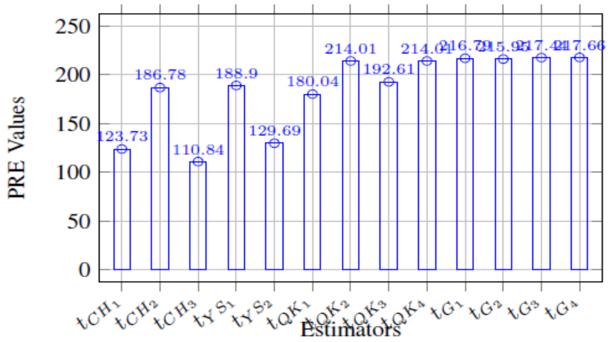

Figure 1. *PRE* values for sample size 130 of the simulation study.

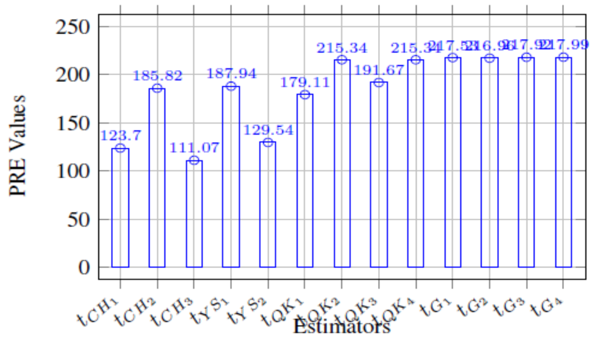

Figure 2. *PRE* values for sample size 135 of the simulation study.

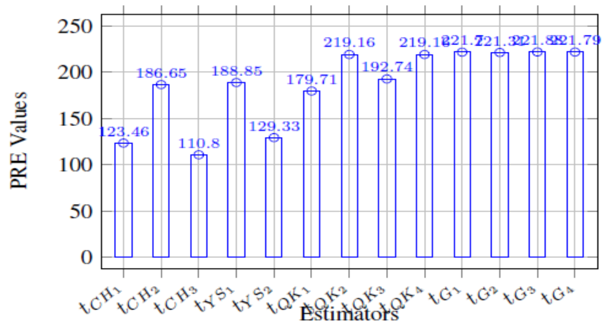

Figure 3. *PRE* values for sample size 140 of the simulation study.

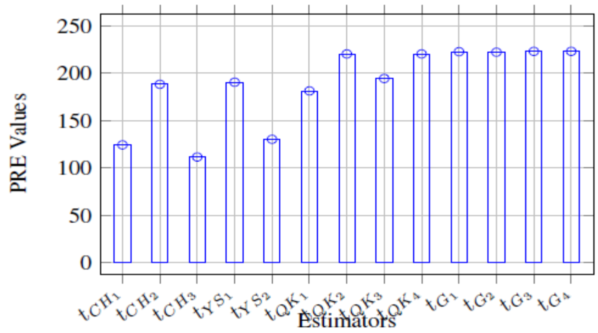

Figure 4. *PRE* values for sample size 144 of the simulation study.

5. Real Data Application

In this section, we apply the Adaptive Cluster Sampling (ACS) design utilizing the transformed population approach developed by Dryver and Chao (2007). This approach is used to evaluate the performance of various estimators, including both the newly developed estimators and the existing competing ones outlined in Table 2, in estimating the average number of thorny plants using the dataset from Latpate & Kshirsagar (2020). Latpate & Kshirsagar (2020) identified a negative correlation between the presence of aluminum in the soil and the occurrence of thorny plants. According to their findings, higher aluminum content in the soil is associated with fewer thorny plants. Based on this observation, we use the percentage of aluminum as the auxiliary variable to estimate the average number of thorny plants in the study area.

In the context of the ACS design under investigation, we define the sample criteria $C=\{y_i>0\}$, where y_i represents the number of thorny plants in the i^{th} randomly selected quadrat (as described by Turk & Borkowski, 2005). This condition stipulates that only quadrats with a positive count of thorny plants are considered. If this condition is met, the surrounding areas, based on the first-order neighborhood concept (Qureshi *et al.*, 2018), will be examined for further analysis. From a total population size of N=400, we draw a sample of n=140 quadrats. We then apply all the estimators, both newly developed and those from previous studies (see Table-2), to estimate the average number of thorny plants. The performance of these estimators, as well as a detailed description of the dataset, are presented and summarized in the table below. This comprehensive evaluation helps to assess how effectively each estimator performs in estimating the average number of thorny plants under the ACS design.

Table 4. *MSE* and *PRE* of all the estimators in estimating average number of thorny plants

Description of population	Estimators	*MSEs*	*PREs*
$C_{w_y}^2$=6.36	t_{TH}	5.5968	100.00
$C_{w_x}^2$=0.16	t_{DC}	6.9780	80.21
$\beta_1(w_x)= -0.3153$	t_{CH_1}	6.9613	80.40
$\beta_2(w_x)= -0.9276$	t_{CH_2}	6.9964	80.00
$S_{w_y}^2$=1205.48	t_{CH_3}	7.0181	79.75
$S_{w_x}^2$=210.16	t_{YS_1}	6.9638	80.37
$\rho_{w_y w_x}$=−0.6974	t_{YS_2}	6.8643	81.53
N=400	t_{QK_1}	6.9784	80.20
n=140	t_{QK_2}	6.9300	80.76
MR=34.02	t_{QK_3}	6.9783	80.20
HL=36.64	t_{QK_4}	6.9400	80.65
TM=37.6	t_G	2.8746	194.70

6. Results and Discussion

The objective of this paper is to investigate estimators that can be applied regardless of the correlation between the auxiliary variable and the survey variable. These estimators, known as ratio-cum-product type estimators, have not been extensively studied in the context of ACS (Adaptive Cluster Sampling) design. To address this gap in knowledge, this study proposes a new novel type class of RCP (ratio cum product) type exponential estimators utilizing one auxiliary variable. This involves integrating the ratio-cum-product type structure with an exponential function, drawing inspiration from previous works by Singh & Espejo (2003), Singh *et al.* (2016), and Dansawad (2023). We then developed several novel estimators from this proposed class using various known parameters of the auxiliary variable, including the coefficient of skewness, kurtosis, mid-range, and Hodges-Lehmann estimator.

To evaluate the performance of these new estimators, we derived expressions of their mean square error (MSE) and bias up to the first order of approximation. We found that the minimum MSE for the proposed class was equivalent to that of the minimum MSE achieved by the regression estimator discussed by Chutiman (2013). We compared the efficiency of the newly developed ratio-cum-product type exponential estimators against existing related estimators, as outlined in Table 2, through a theoretical MSE comparison presented in Section 3. To support this theoretical analysis, we conducted a simulation study. The results of this simulation study, detailed in Table 3, demonstrated that all the new estimators—$t_{G_1}, t_{G_2}, t_{G_3}$ and t_{G_4}, —provided higher percentage relative efficiency (PRE) compared to the competing estimators discussed in the paper. Additionally, we applied all these estimators to estimate the average number of thorny plants using data from Latpate & Kshirsagar (2020). The results from this real data application, presented in Table 4, showed that the proposed class of ratio-cum-product type exponential estimators yielded higher PREs compared to the existing competing estimators.

Based on this thorough analysis, we conclude that the newly developed ratio-cum-product type estimators are particularly effective for populations that are rare or hard to access, which is the specific scenario for which the ACS design was created by Thompson (1990). Future research should focus on applying ACS design to datasets related to disease that resemble ACS-type populations and further evaluating the performance of all existing estimators in such contexts.

Conflicts of Interest

The authors declare no conflict of interest.

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