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ARTICLE

A new extended Remkan distribution and its application to the cancer data

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Abstract
A new two-parameter Remkan distribution is derived. In this study a new version of the length-biased Remkan distribution is discussed. Mathematical properties of the new distribution, including moments, moment-generating functions, shape, order statistics, Renyi entropy, and Bonferroni curves, have been proposed. Also discussed was the simulation study of the proposed distribution. The length-biased Remkan distribution in survival data is discussed on real lifetime data from engineering and medical science. Finally analyzed, a real-life data set is fitted and the fit has been found to be good.

Keywords: Length biased; Remkan distribution; Order Statistics; Maximum Likelihood Estimation; Bonferroni and Lorenz Curve.

1. Introduction

There are many lifetime distributions available for survival analysis such as Samade (Aderoju, 2021), Hanza et al (Aijaz *et al.*, 2020), Lomax (Ahmad *et al.*, 2016), Iwueze (Elechi *et al.*, 2022), Inverse-Gaussian (Khatree, 1989), Maxwell (Modi & Gill, 2015), Odoma (Odoma & Ijomah, 2019), Prakaamy (Shukla, 2018a), Ram (Shuka, 2018b), and Pranav (Umeh & Ibenegbu, 2019) to name but a few.

The length-biased Remkan distribution contributes to a deeper understanding of lifetime distributions (also known as survival distributions) and their applications. In this distribution, various properties, moments, and likelihood estimation have been studied.

The Probability Density Function of Remkan Distribution is given by

$$f(x;\eta,\phi) = \frac{\eta^2}{(\eta+2\phi+6)}\left[1+\phi\eta x^2+\eta^2 x^3\right] e^{-\eta x}; x>0, \eta>0, \phi>0. \tag{1}$$

The Cumulative Distribution Function of Remkan Distribution is given by

$$F(x;\eta,\phi) = 1-\left[1+\frac{\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}. \tag{2}$$

Recently, Uwaeme & Akpan (2024) discussed the Remkan distribution with applications to real-life data, which shows more flexibility than traditional distribution such as Gamma, Weibull, Power Lindly, Quasi Lindly, and Exponential. In this paper, a new two-parameter lifetime distribution has been introduced: the Remkan distribution, newly introduced to the length-biased Remkan distribution.

2.The Length Biased Remkan Distribution (LBRD)

The probability density function of the length biased Remkan Distribution is given by

$$f_w(x) = \frac{w(x)f(x)}{E\big(w(x)\big)}; x > 0, \tag{3}$$

$$f_w(x) = \frac{xf(x)}{E(\mathrm{x})}; x > 0$$

where $w(x)$ be a non-negative weight function and $E\big(w(x)\big) = \int w(x)f(x)dx < \infty$.

$$f_w(x) = \frac{xf(x)}{E(x^2)}; x > 0$$

where,

$$E(x) = \int_0^\infty x\, f(x;\eta,\phi)dx$$

$$= \int_0^\infty x \frac{\eta^2}{(\eta+2\phi+6)}\left[1 + \phi\eta x^2 + \eta^2 x^3\right] e^{-\eta x}\, dx$$

$$= \frac{\eta^2}{(\eta+2\phi+6)} \int_0^\infty x \left[1 + \phi\eta x^2 + \eta^2 x^3\right] e^{-\eta x}\, dx$$

$$= \frac{\eta^2}{(\eta+2\phi+6)} \int_0^\infty x\, e^{-\eta x}\, dx\ + \phi\eta \int_0^\infty x^3 e^{-\eta x}\, dx\ + \eta^2 \int_0^\infty x^4 e^{-\eta x}\, dx$$

$$= \frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{\eta^3\ \Gamma 2}{\eta^2} + \frac{\phi\eta^2\ \Gamma 4}{\eta^4} + \frac{\eta^2\ \Gamma 5}{\eta^5}\right]$$

$$= \frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{\eta^3\ \Gamma 2+\phi\eta^2\ \Gamma 4+\eta^2\ \Gamma 5}{\eta^5}\right]$$

$$= \frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{\eta^3+\phi\eta^2 6+\eta^2 24}{\eta^5}\right]$$

$$= \frac{\eta^2(\eta+6\phi+24)}{\eta^3(\eta+2\phi+6)}$$

$$E(x) = \frac{\eta+6\phi+24}{\eta(\eta+2\phi+6)} \tag{4}$$

Substitute (1) and (4) in equation (3), we will get the required probability density function of length biased Remkan distribution as

$$= \frac{\frac{\eta^2}{(\eta+2\phi+6)}}{\frac{\eta+6\phi+24}{\eta(\eta+2\phi+6)}}$$

$$f_l(x) = \frac{\eta^3}{\eta+6\phi+24} x(1 + \phi\eta x^2 + \eta^2 x^3)e^{-\eta x}\, dx \tag{5}$$

Now, the cumulative distribution function (cdf) of the length-biased Remkan distribution (LBRD) is obtained as

$$F_l(x) = \int_0^x f_w(x)dx \tag{6}$$

$$F_l(x) = \frac{\eta^3}{(\eta+6\phi+24)} x[1 + \phi\eta x^2 + \eta^2 x^3]\, e^{-\eta x}\, dx$$

$$= \frac{\eta^3}{(\eta+6\phi+24)} \int_0^t x + \phi\eta x^3 + \eta^2 x^4\, e^{-\eta x}\, dx$$

$$x = \frac{t}{\eta}, \quad \eta x = t, \quad dx = \frac{1}{\eta} dt$$

$$\frac{\eta^3}{(\eta+6\phi+24)} = \int_0^{\eta x} x\, e^{-\eta x}\, dx + \phi\eta \int_0^{\eta x} x^3\, e^{-\eta x}\, dx + \eta^2 \int_0^{\eta x} x^4\, e^{-\eta x}\, dx$$

$$\frac{\eta^3}{(\eta+6\phi+24)} = \int_0^{\eta x} \left(\frac{t}{\eta}\right) e^{-t}\, \frac{1}{\eta}\, dt + \phi\eta \int_0^{\eta x} \left(\frac{t}{\eta}\right)^3 e^{-t} \frac{1}{\eta}\, dt + \eta^2 \int_0^{\eta x} \left(\frac{t}{\eta}\right)^4 e^{-t} \frac{1}{\eta}\, dt$$

$$\frac{\eta^3}{(\eta+6\phi+24)} = \int_0^{\eta x} \left(\frac{t}{\eta^2}\right) + \phi\eta \left(\frac{t}{\eta^4}\right)^3 + \eta^2 \left(\frac{t}{\eta^5}\right)^4$$

$$\frac{\eta^3}{(\eta+6\phi+24)} = \frac{\eta^3}{\eta^5} \big(\gamma(2, \eta x) + \phi\eta^2\gamma(4, \eta x) + \eta^2(5, \eta x)\big)$$

$$\frac{\eta^3}{(\eta+6\phi+24)} = \frac{1}{\eta^5} \big(\eta^3\gamma(2, \eta x) + \phi\eta^2\gamma(4, \eta x) + \eta^2(5, \eta x)\big)$$

$$= \frac{\eta^3\gamma(2,\eta x)+\phi\eta^2\gamma(4,\eta x)+\eta^2(5,\eta x)}{\eta^2(\eta+6\phi+24)}$$

$$= \frac{\eta^2(\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x))}{\eta^2(\eta+6\phi+24)}$$

$$F_l(x) = \frac{\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{\eta+6\phi+24} \tag{7}$$

Plots of the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the Length Biased Remkan Distribution Plots for various parameter sets are presented in Figures 1 and 2, respectively.

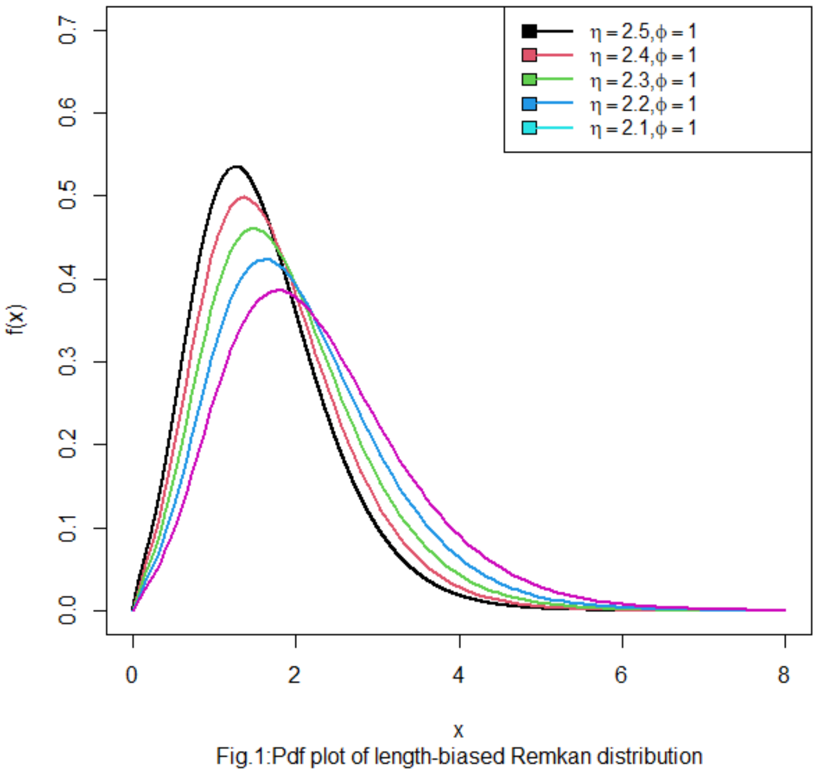

Figure 1. Probability Density Function (PDF) of the Length Biased Remkan Distribution Plots for various parameter sets.

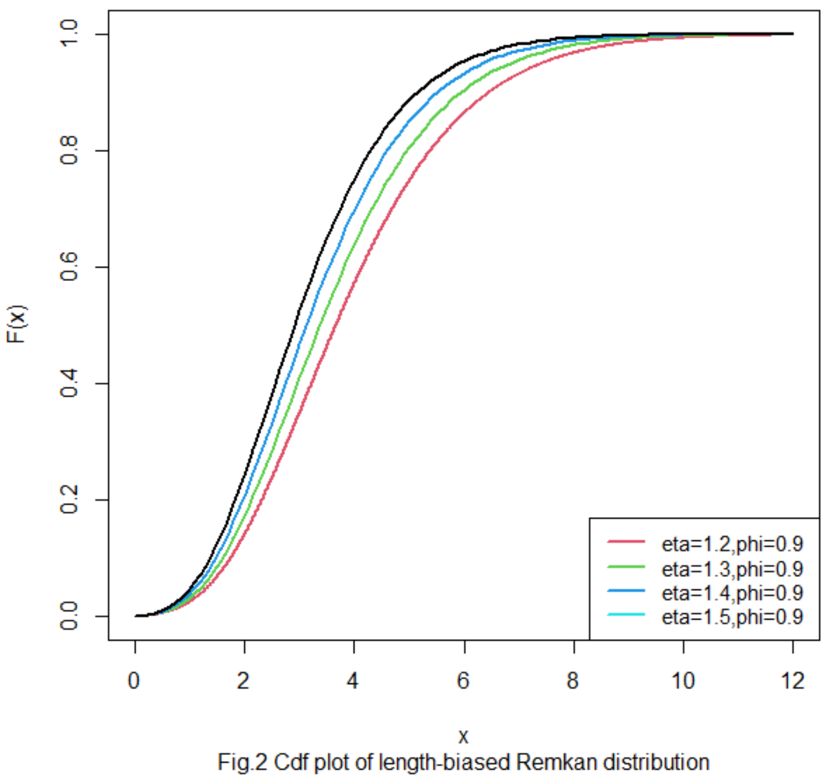

Figure 2. Cumulative Distribution Functions (CDF) of the Length Biased Remkan Distribution for various parameter sets.

3.Reliability Analysis

The above section has been provide the survival function, hazard rate function, reverse hazard rate function, odds rate function, cumulative hazard rate function, and mills ratio for the specified Length biased Remkan Distribution

3.1 Survival Function

The survival function of the length-biased Remkan distribution is given by

$$S(x) = 1 - F_l(x;\eta,\phi)$$

$$S(x) = 1 - \left(\frac{\eta x(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{(\eta+6\phi+24)}\right) \quad (8)$$

3.2 Hazard Rate Function

The hazard function of the length-biased Remkan distribution is given by

$$h(x) = \frac{f_l(x;\eta,\phi)}{1-F_l(x;\eta,\phi)}$$

$$h(x) = \frac{\frac{\eta^3}{\eta+6\phi+24}x\left(1+\phi\eta x^2+\eta^2x^3\right)e^{-\eta x}\,dx}{1\left(\frac{\eta x(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{(\eta+6\phi+24)}\right)}$$

$$h(x) = \left(\frac{\eta^3}{(\eta+6\phi+24)-(\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x))}x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}\right) \quad (9)$$

Figures 3 and 4 present the survival and harzard functions of the Length Biased Remkan distribution for various parameter sets.

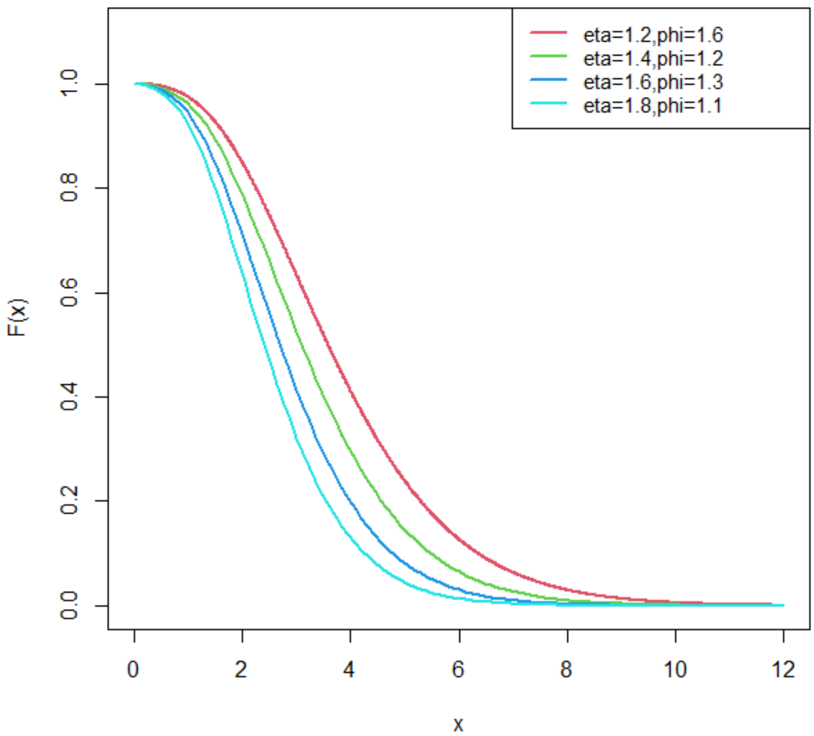

Fig.3 survival function of length-biased Remkan distribution

Figure 3. Survival Functions of the Length Biased Remkan distribution for various parameter Sets.

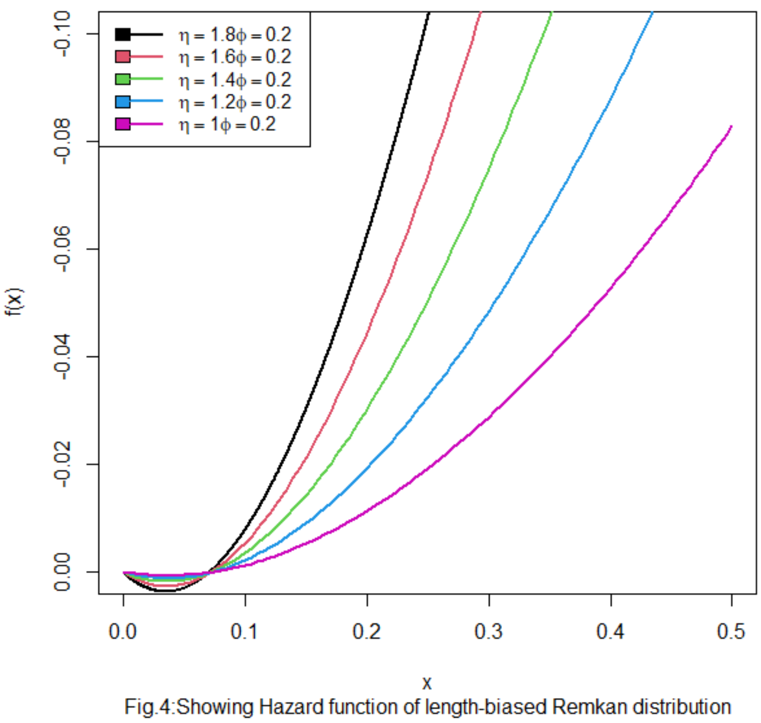

Figure 4. Hazard Functions of the Length Biased Remkan distribution for various parameter sets.

3.3 Reverse hazard Rate Function

The Reverse hazard rate of length-biased Remkan distribution is given by

$$h_r(x) = \frac{f_l(\eta;\phi,\theta)}{F_l(x;\eta,\phi)}$$

$$h_r(x) = \left(\frac{\eta^3}{(\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x))}x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}\right) \quad (10)$$

3.4 Odds Rate Function

Odds Rate function of length-biased Remkan distribution is given by

$$O(x) = \frac{F_l(x;\eta,\phi)}{1-F_l(x;\eta,\phi)}$$

$$O(x) = \left(\frac{\frac{\eta x(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{(\eta+6\phi+24)}}{1-\left(\frac{\eta x(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{(\eta+6\phi+24)}\right)}\right)$$

$$O(x) = \left(\frac{\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{(\eta+6\phi+24\,)-\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}\right) \quad (11)$$

3.5 Cumulative Hazard Rate Function

Cumulative hazard rate function of length-biased Remkan distribution is given by

$$H(x) = -\ln(1-F_l(x)$$

$$H(x) = -\ln\,(1-F_l(x;\eta,\phi)$$

$$H(x) = -\ln\left(1-\frac{\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{(\eta+6\phi+24)}\right) \quad (12)$$

3.6 Mills Ratio

Mills Ratio of the length-biased Remkan distribution is

$$\text{Mills Ratio} = \frac{1}{h_r(x)}$$

$$\text{Mills Ratio} \left(\frac{\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)}{\eta^3 x(1+\phi\eta x^2+\eta^2 x^3)e^{-\eta x}}\right) \tag{13}$$

4. Moments and Associated Measures

Let X denotes the random variable of length-biased Remkan distribution with parameter η and ϕ then the r^{th} order moments $E(X^r)$ of length-biased Remkan distribution is obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r \, f_l(x) dx$$

$$= \int_0^\infty x^r \frac{\eta^3}{(\eta+6\phi+24)} x[\,1 + \phi\eta x^2 + \eta^2 x^3]\, e^{-\eta x}\, dx$$

$$= \frac{\eta^3}{(\eta+6\phi+24)} \int_0^\infty x^{r+1} \,[\,1 + \phi\eta x^2 + \eta^2 x^3]\, e^{-\eta x}\, dx$$

$$= \frac{\eta^3}{(\eta+6\phi+24)} \int_0^\infty x^{r+1}\, e^{-\eta x}\, dx + \phi\eta \int_0^\infty x^{r+3}\, e^{-\eta x}\, dx + \eta^2 \int_0^\infty x^{r+4}\, e^{-\eta x}\, dx$$

$$= \frac{\eta^3}{(\eta+6\phi+24)} \left(\frac{\lceil r+2}{\eta^{r+2}} + \frac{\phi\eta \lceil r+4}{\eta^{r+4}} + \frac{\eta^2 \lceil r+5}{\eta^{r+5}}\right)$$

$$= \frac{\eta^3}{(\eta+6\phi+24)} \left[\frac{\eta^3 \lceil r+2+\phi\eta^2 \lceil r+4+\eta^2 \lceil r+5}{\eta^{r+5}}\right]$$

$$= \frac{\eta^3}{(\eta+6\phi+24)} \eta^2 \left[\frac{\eta \lceil r+2+\phi \lceil r+4+ \lceil r+5}{\eta^{r+5}}\right]$$

$$= \frac{\eta^3}{(\eta+6\phi+24)} \left[\frac{\eta \lceil r+2+\phi \lceil r+4+ \lceil r+5}{\eta^{r+3}}\right]$$

$$= \mu_r' = \frac{\eta \lceil r+2+\phi \lceil r+4+ \lceil r+5}{\eta^r(\eta+6\phi+24)} \tag{14}$$

Put $r = 1, 2, \ldots$ in equation we will obtain the first Raw moments of length-biased Remkan distribution which is given by

$$E(X^1) = \mu_1' = \frac{\eta \lceil 1+2+\phi \lceil 1+4+ \lceil 1+5}{\eta^r(\eta+6\phi+24)}$$

$$\mu_1' = \frac{2\eta+24\phi+120}{\eta^r(\eta+6\phi+24)}$$

$$E(X^2) = \mu_2' = \frac{\eta \lceil 2+2+\phi \lceil 2+4+ \lceil 2+5}{\eta^r(\eta+6\phi+24)}$$

$$\mu_2' = \frac{6\eta+120\phi+720}{\eta^r(\eta+6\phi+24)}$$

4.1 Variance

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \left(\frac{6\eta+120\phi+720}{\eta^r(\eta+6\phi+24)} - \left(\frac{2\eta+24\phi+120}{\eta^r(\eta+6\phi+24)}\right)^2\right)$$

$$= \left(\frac{6\eta+120\phi+720}{\eta^r(\eta+6\phi+24)} - \frac{(2\eta+24\phi+120)^2}{\eta^r(\eta+6\phi+24)^2}\right) \tag{15}$$

5. Hormonic Mean

The Harmonic mean of the proposed model can be obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x)dx$$

$$= \int_0^\infty \frac{1}{x}\frac{\eta^3}{\eta+6\phi+24} x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}dx$$

$$= \frac{\eta^3}{\eta+6\phi+24}\frac{1}{x} x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}dx$$

$$= \frac{\eta^3}{\eta+6\phi+24}\int_0^\infty \frac{1}{x} x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}dx$$

$$= \frac{\eta^3}{\eta+6\phi+24}\int_0^\infty (1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}dx$$

$$= \frac{\eta^3}{\eta+6\phi+24}\int_0^\infty e^{-\eta x}\, dx + \phi\eta\int_0^\infty x^2\, e^{-\eta x}\, dx + \eta^2\int_0^\infty x^3\, e^{-\eta x}\, dx$$

$$= \frac{\eta^3}{\eta+6\phi+24}\left(\frac{1}{\eta}+\phi\eta\,\frac{\Gamma 3}{\eta^3}+\eta^2\,\frac{\Gamma 4}{\eta^4}\right)$$

$$= \frac{\eta^3}{\eta+6\phi+24}\left(\frac{\eta^3+\phi\eta^2\,\Gamma 3+\eta^2\,\Gamma 4}{\eta^4}\right)$$

$$= \frac{\eta^3}{\eta+6\phi+24}\left(\frac{\eta^3+\phi\eta^2 2+\eta^2 6}{\eta^4}\right)$$

$$= \frac{\eta^3+\phi\eta^2 2+\eta^2 6}{\eta(\eta+6\phi+24)}$$

$$= \frac{\eta(\eta+2\phi+6)}{(\eta+6\phi+24)} \tag{16}$$

6. Moment Generating Function and Characteristics Function

Suppose the random variable *X* follows Length biased Remkan distribution with parameters η and ϕ, then the MGF of *X* can be obtained as:

$$M_X(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f_l(x;\eta,\phi)dx$$

Using Taylor's Series Expansion

$$M_X(t) = \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right]$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\int_0^\infty x^j g_1(x;\eta,\phi)dx$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\mu_j{}'$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\eta\,\Gamma j+2+\phi\,\Gamma j+4+\Gamma j+5}{\eta^j(\eta+6\phi+24)}\right)$$

Similarly, the Characteristics function of Length-biased Remkan Distribution can be obtained by

$\phi_X(t) = M_X(it)$

$$M_X(it) = \frac{1}{\eta+6\phi+24}\sum_{j=0}^{\infty}\frac{t^j}{j!\eta^j}(\eta\,\Gamma j+2+\phi\,\Gamma j+4+\,\Gamma j+5) \tag{17}$$

7. Order Statistics

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \dots, X_n$ drawn from the continuous population with probability density function $g_x(x)$ and cumulative density function with $G_x(x)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!}f_X(x)[F_X(x)]^{r-1}[1-F_X(x)]^{n-r}$$

The probability density function of r^{th} order statistics $X_{(r)}$ of Length-biased Remkan distribution is given by

$$\frac{n!}{(r-1)!\,(n-r)!}\left(\frac{\eta^3}{\eta+6\phi+24}x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}\right)$$
$$\times\left(\frac{1}{\eta+6\phi+24}\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)\right)^{r-1}$$
$$\times\left(1-\frac{1}{\eta+6\phi+24}\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)\right)^{n-r}$$

Therefore, the Probability density function of first Order Statistics X_1 of Length-biased Remkan distribution is can be obtained as

$$f_{X(1)}(x) = \frac{n(n-1)!}{(1-1)!\,(n-1)!}\left(\frac{\eta^3}{\eta+6\phi+24}x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}\right)$$
$$\times\left(1-\frac{1}{\eta+6\phi+24}\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)\right)^{n-1}$$

$$f_{X(n)}(x) = \frac{n!}{(n-1)!\,(1-1)!}\left(\frac{\eta^3}{\eta+6\phi+24}x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}\right)$$
$$\times\left(\frac{1}{\eta+6\phi+24}\eta\gamma(2,\eta x)+\phi\gamma(4,\eta x)+\gamma(5,\eta x)\right)^{n-1} \tag{18}$$

8. Estimation of Parameters

In this section provides the Maximum likelihood estimates and Fisher information matrix.

8.1 Likelihood Ratio Test

Let $X_1, X2_{,\dots,}X_n$ be a random sample from the length-biased Remkan distribution. To test the hypothesis

$H_0: f(x) = f(x;\eta,\phi)$ against $H_{1:} f(x) = f_l(x;\eta,\phi)$

In order to test whether the random sample of size n comes from the length-biased Remkan distribution, the following test statistics is used by

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_l(x;\eta,\phi)}{f(x;\eta,\phi)}$$

$$= \prod_{i=0}^{n} \left(\frac{\frac{\eta^3}{\eta+6\phi+24} x\left(1+\phi\eta x^2+\eta^2 x^3\right)e^{-\eta x}}{\frac{\eta^2}{\eta+2\phi+6}\left(1+\phi\eta x^2+\eta^2 x^3\right)e^{-\eta x}} \right)$$

$$= \prod_{i=0}^{n} \frac{\eta(\eta+2\phi+6)}{\eta+6\phi+24} x_i$$

$$= \left(\frac{\eta(\eta+2\phi+6)}{\eta+6\phi+24}\right)^n \prod_{i=0}^{n} x_i$$

We should reject the null hypothesis, if

$$\Delta = \left(\frac{\eta(\eta+2\phi+6)}{\eta+6\phi+24}\right)^n \prod_{i=0}^{n} x_i > k \text{ (or)}$$

Equivalently, we shall reject the null hypothesis, if

$$\Delta^* = \prod_{i=0}^{n} x_i > k \left(\frac{\eta(\eta+2\phi+6)}{\eta+6\phi+24}\right)^n$$

$$\Delta^* = \prod_{i=0}^{n} x_i > k^* \; where,$$

$$k^* = k \left(\frac{\eta(\eta+2\phi+6)}{\eta+6\phi+24}\right)^n \tag{19}$$

Then $p(\Delta^* > \lambda^*)$, where, $\lambda^* = \prod_{i=0}^{n} x_i$ is less than a specified level of significance, and $\prod_{i=0}^{n} x_i$ is the observed value of Δ^*

8.2 Maximum Likelihood Estimate and Fisher's Information Measure

$$L(x) = \prod_{i=1}^{n} f_l\,(x)$$

$$L(x) = \prod_{i=1}^{n} \frac{\eta^3}{\eta+6\phi+24} x_i\left(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3\right)e^{-\eta x}$$

$$L(x) = \frac{\eta^{3n}}{(\eta+6\phi+24)^n} \prod_{i=1}^{n} x_i\left(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3\right)e^{-\eta x} \tag{20}$$

The log likelihood function is given by

$$= n\,log(\eta^3) - n\log(\eta+6\phi+24) + 2\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n} log\left(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3\right) - \eta\sum_{i=1}^{n} x_i$$

Now, differentiating the log likelihood equation (20)

$$\frac{\partial logL}{\partial\eta} = n\left(\frac{3\eta^4}{\eta^3}\right) - n\frac{1}{(\eta+6\phi+24)} + \sum_{i=1}^{n} \frac{\left(\phi {x_i}^2+2\eta {x_i}^3\right)}{\left(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3\right)} - \sum_{i=1}^{n} x_i = 0 \tag{21}$$

$$\frac{\partial logL}{\partial \phi} = -n\frac{6}{(\eta+6\phi+24)} + \sum_{i=1}^{n}\frac{(\eta {x_i}^2)}{(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3)} = 0 \quad (22)$$

For the purpose of obtaining the confidence interval we use the asymptotic normality results. We have that if $\hat{\lambda} = (\hat{\eta}, \hat{\phi})$ denotes the MLE of $\lambda = (\eta, \phi)$ We can state the results as follows

$$\sqrt{n}\,(\hat{\lambda} - \lambda) \rightarrow N_2(0, I^{-1}(\lambda))$$

where, $I(\lambda)$ is Fisher's Information Matrix. i.e.,

$$I(\lambda) = -\frac{1}{n}\begin{bmatrix} E\left[\frac{\partial^2 logL}{\partial \eta^2}\right] & E\left[\frac{\partial^2 logL}{\partial \eta \partial \phi}\right] \\ E\left[\frac{\partial^2 logL}{\partial \phi \partial \eta}\right] & E\left[\frac{\partial^2 logL}{\partial \phi^2}\right] \end{bmatrix}$$

$$\left[\frac{\partial^2 logL}{\partial \eta^2}\right] = E\left[\frac{\partial}{\partial \eta}\left(\frac{\partial logL}{\partial \eta}\right)\right]$$

$$\left[\frac{\partial^2 logL}{\partial \eta^2}\right] = n\left(\frac{6\eta^4 - 9\eta^4}{\eta^6}\right) - n\frac{1}{(\eta+6\phi+24)^2} + \frac{2{x_i}^3 + 2\phi\eta {x_i}^5 + 2\eta^2 {x_i}^6 - 2\phi {x_i}^2 + 2\eta {x_i}^3}{(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3)^2}$$

$$\left[\frac{\partial^2 logL}{\partial \phi^2}\right] = E\left[\frac{\partial}{\partial \phi}\left(\frac{\partial logL}{\partial \phi}\right)\right]$$

$$\left[\frac{\partial^2 logL}{\partial \phi^2}\right] = -n\frac{-36}{(\eta+6\phi+24)^2} - \frac{-\eta^2 {x_i}^4}{\left(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3\right)^2} \quad (23)$$

$$\left[\frac{\partial^2 logL}{\partial \eta \partial \phi}\right] = -n\frac{-6}{(\eta+6\phi+24)^2} + \frac{{x_i}^2 + 2\eta^4 {x_i}^{10}}{\left(1+\phi\eta {x_i}^2+\eta^2 {x_i}^3\right)^2} \quad (24)$$

9. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are used in economics in order to study the income, etc,. but it is used in other fields like demography, insurance, medicine and reliability. The Bonferroni and Lorenz curves are given by equations (25) and (26).

$$B(p) = \frac{1}{p\mu_1{}'}\int_0^q xf(x)dx \quad (25)$$

$$B(p) = \frac{1}{p\mu_1{}'}\int_0^q xf(x;\eta,\phi)dx$$

and

$$L(p) = \frac{1}{\mu_1{}'}\int_0^q xf(x;\eta,\phi)dx$$

where, $q = F^{-1}(p)$; $q\epsilon[0,1]$ and $\mu = (x)$

Hence, the Bonferroni and Lorenz curves of our distribution are given by,

$$\mu = \frac{2\eta+24\phi+120}{\eta(\eta+6\phi+24)}$$

$$B(p) = \frac{\eta(\eta+6\phi+24)}{2\eta+24\phi+120} \int_0^q \frac{\eta^3}{\eta+6\phi+24} x\left(1+\phi\eta x_i^{\,2}+\eta^2 x_i^{\,3}\right)e^{-\eta x}dx$$

$$= \frac{\eta^4}{2\eta+24\phi+120} \int_0^q x\left(1+\phi\eta x_i^{\,2}+\eta^2 x_i^{\,3}\right)e^{-\eta x}dx$$

$$= \frac{\eta^4}{2\eta+24\phi+120} \int_0^q x e^{-\eta x}dx + \phi\eta \int_0^q x^3 e^{-\eta x}dx + \eta^2 \int_0^q x^4 e^{-\eta x}dx$$

Put, $x = \frac{t}{\eta}$, $\eta x = t$, $dx = \frac{1}{\eta}dt$

when $x \to 0, t \to 0$, and $x \to x, t \to \eta x$

$$= \frac{\eta^4}{2\eta+24\phi+120} \frac{1}{\eta^2} \int_0^{\eta q} t\, e^{-t}dt + \frac{1}{\phi\eta^5} \int_0^{\eta q} t^3 e^{-t}dt + \frac{1}{\eta^7} \int_0^{\eta q} t^4 e^{-t}dt$$

$$= \frac{\eta^4}{2\eta+24\phi+120} \frac{\eta}{\eta^2} \int_0^{\eta q} t^{2-1}e^{-t}dt + \frac{\phi}{\eta^3} \int_0^{\eta q} t^{4-1}e^{-t}dt + \frac{1}{\eta^3} \int_0^{\eta q} t^{5-1}e^{-t}dt$$

After the simplification we get

$$B(p) = \frac{\eta\left(\gamma(3,\eta q)+\phi\gamma(4,\eta q)+\gamma(5,\eta q)\right)}{2\eta+24\phi+120} \tag{26}$$

10. Entropies

Entropy is important in different areas such as probability and economics, communication theory, physics, and statistics. Entropies are applied to quantify a system's diversity, uncertainty, or randomness. An indicator of the uncertainty's variation is the entropy of a random variable X.

10.1 Renyi Entropies

The Renyi entropy is significant as a diversity index. The Renyi entropy is also important in quantum information. It can be used as a measure of entanglement for a given probability distribution. Renyi entropy is given by

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(f(x)\right)^\lambda dx\,;\ \lambda > 0, \lambda \neq 1$$

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(f(x;\eta,\phi)\right)^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty \left(\frac{\eta^3}{\eta+6\phi+24} x(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}\right)^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \left(\frac{\eta^3}{\eta+6\phi+24}\right)^\lambda \int_0^\infty x^\lambda (1+\phi\eta x^2+\eta^2x^3)^\lambda\, e^{-\lambda\eta x}dx$$

Using binomial expansion

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} 1^{\lambda-i} (\phi\eta x^2+\eta^2x^3)^i$$

$$R_\lambda = \frac{1}{1-\lambda} log \left(\frac{\eta^3}{\eta+6\phi+24}\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} (\phi\eta)^{i-j}\eta^{2j} \int_0^\infty x^{(\lambda+2i+j+1)-1} e^{-\lambda\eta x}dx$$

$$R_\lambda = \frac{1}{1-\lambda} log \left(\frac{\eta^3}{\eta+6\phi+24}\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i} \binom{\lambda}{i}\binom{i}{j} (\phi\eta)^{i-j}\eta^{2j} \frac{\Gamma\lambda+2i+j+1}{(\eta\lambda)^{\lambda+2i+j+1}} \tag{27}$$

10.2. Tsallis Entropies

The Boltzmann-Gibbs (B-G) statistical property generalization initiated by Tsallis has received a great deal of attention. This B-G statistic was first introduced as the mathematical expansion of Tsallis entropy for continuous random variables; this generalization of B-G was introduced in order to suggest. It is defined as

$$T_\lambda = \frac{1}{\lambda-1}\left(1 - \int_0^\infty \left(f(x;\eta,\phi)\right)^\lambda dx\right) \lambda > 0, \lambda \neq 1$$

$$T_\lambda = \frac{1}{\lambda-1}\left(1 - \int_0^\infty \left(\frac{\eta^3}{\eta+6\phi+24}\right) x(1+\phi\eta x^2+\eta^2 x^3)\, e^{-\eta x}\right)^\lambda dx$$

$$T_\lambda = \frac{1}{\lambda-1}\left(1 - \left(\frac{\eta^3}{\eta+6\phi+24}\right)^\lambda \int_0^\infty x^\lambda (1+\phi\eta x^2+\eta^2 x^3)^\lambda\, e^{-\lambda\eta x} dx\right)$$

Using binomial expansion

$$T_\lambda = \frac{1}{\lambda-1}\left(\frac{\eta^3}{\eta+6\phi+24}\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}(\phi\eta)^{i-j}\eta^{2j}\int_0^\infty x^{(\lambda+2i+j+1)-1}e^{-\lambda\eta x}dx$$

$$T_\lambda = \frac{1}{\lambda-1}\left(\frac{\eta^3}{\eta+6\phi+24}\right)^\lambda \sum_{i=1}^{\lambda}\sum_{j=0}^{i}\binom{\lambda}{i}\binom{i}{j}(\phi\eta)^{i-j}\eta^{2j}\frac{\Gamma\lambda+2i+j+1}{(\eta\lambda)^{\lambda+2i+j+1}} \quad (28)$$

12. Results and Discussion

We have introduced in the present study a new version of the Remkan distribution called a length-biased Remkan distribution that has two parameters. In this study, the Remkan distribution is provided as a newly named distribution called the length-biased Remkan distribution for the lifetime data sets from engineering and medical science. Length-biased Remkan distribution in various properties includes moments and associated measures, hazard rate and survival function, order statistics, Bonferroni and Lorenz curves, and entropy measures have been proposed. Many statistical measures have been proposed for the MLE estimators. The length-biased Remkan distribution's reliability behavior and hazard rate function make it a suitable lifetime model. For the estimating parameter, methods of moments and maximum likelihood estimation have been discussed.

The new length-biased Remkan distribution is fitted to the cancer data and compared with the Remkan distribution.

We consider the following data set of 19 patients suffering from leukaemia blood cancer (non-censored data): (1.013, 1.034, 1.109, 1.226, 1.509, 1.533, 1.563, 1.716, 1.929, 1.965, 2.061, 2.344, 2.546, 2.626, 2.778, 2.951, 3.413, 4.118, 5.136)

In order to compare the performance of length-biased Remkan distribution with Remkan distribution, we are using the criteria values, like AIC (Akaike information criterion), $AICC$ (corrected Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser values of $AIC, AICC, BIC$ and $-2\log L$ can be evaluated by using the formulas as follows:

$$AIC = 2K - 2\log L \quad AICC = AIC + \frac{2k(k+1)}{(n-k-1)} \text{ and } BIC = k\log n - 2\log L$$

where $K =$ number of parameters, n samplesize and $-2\log L$ is the maximized value of loglikelihood function. Results are presented in Table 1.

Table 1. MLEs, AIC, BIC, AICC, and $-2\log L$ of the fitted distribution for the given data set

Distribution	ML Estimates	-2logL	AIC	BIC	AICC
Length-biased Remkan Distribution	$\hat{\eta} = 1.785410e + 00\ (2.048014e - 01)$ $\hat{\phi} = 1.328847e + 04\ (2.372658e + 04)$	51.77355	55.77355	57.66243	56.52355
Remkan Distribution	$\hat{\eta} = 1.399119e + 00\ (1.876752e - 01)$ $\hat{\phi} = 1.727931e + 04\ (3.355444e + 04$	53.58735	57.58735	59.47622	58.33735

From the table 1, it can be observed that the length-biased Remkan distribution has lower -2L*, AIC, BIC, AICC, and -2logL than the Remkan distribution. Thus, the length-biased Remkan distribution gives better performance than the Remkan distribution.

13. Conclusions

The novel length-biased Remkan distribution gives better performance than the Remkan distribution when fitted to the cancer data.

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Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

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