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ARTICLE

Concomitants of lower record statistics for Bivariate Pseudo Powered Inverse Rayleigh distribution

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Abstract

The concept of p- th record value was introduced by Dziubdziela and Kopocinski (1976). In this paper, we have considered concomitants of p- th lower record values (LRV) for Bivariate Pseudo Powered Inverse Rayleigh ($BPPIR$) distribution. Further single and joint distribution of concomitants of p- th LRV are obtained and expressions for single and product moments are derived. Additionally, we provide the minimum variance linear unbiased estimator of the location and scale parameters of the concomitants of p-th lower record values. The tables for survival function and hazard function has also been obtained. The explicit expression for the reliability is also derived for $BPPIR$ distribution.

Keywords: Concomitants, Lower record values, Moments, Minimum variance linear unbiased estimator, Reliability, Bivariate Pseudo Powered Inverse Rayleigh distribution.

1. Introduction

An observation is called a record if its value is greater (or lesser) than all the previous observations Chandler (1952). Record values (RV) have been effectively applied in numerous real-world scenarios as well as in scientific domains such as weather, education, economics studies, sports data, industrial stress testing, meteorology analysis, epidemiology, and Olympic and COVID-19 records. Prediction about future record values are very important such as intensity of the next earthquake,

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highest(record) level of water dam hold or discharge, extreme point of market index. For a survey on important results in this area one may refer to Asgharzed & Abdi (2012), Khan & Zia (2013), Minimol & Thomas (2013, 2014), Bdair & Raqab (2014), Khan *et al.* (2015), Khan & Khan (2016), Khan *et al.* (2017), Singh & Khan (2018), Kumar & Dey (2018), Singh *et al.* (2020) are just a few of the authors who have recently used the idea of "RV" in their writing.

Let $\{W_a,\ a \geq 1\}$ be a sequence of independent and identically distributed (*iid*) continuous random variables with distribution function (*df*) $F(w)$ and probability distribution function (*pdf*) $f(w)$. For a fixed positive integer p, we define the sequence $\{L_p(a), a \geq 1\}$ of p-th lower record times of $\{W_a,\ a \geq 1\}$ as follows:

$$L_1^{(p)} = 1,$$

$$L_{a+1}^{(p)} = \min\{j > L_p(a) : W_{p:L_p(a)+p-1} > W_{p:j+p-1}\}.$$

The sequence $\{V_a^{(p)}\ , a \geq 1\}$, where $V_a^{(p)} = W_{p:L_p(a)+k-1}$, $a = 1,2,...,$ is called the sequence p-th LRV of $\{W_a\ , a \geq 1\}$. Suppose $V_0^{(p)} = 0$, then for $p = 1$, we have $V_a^{(1)} = W_{L_a,}$ $a \geq 1$, i.e. the LRV. The pdf p-th LRV $V_a^{(p)}$ and joint pdf of $V_b^{(p)}$ and $V_a^{(p)}$ p-th LRV are defined as

$$f_{V_a^{(p)}}(w) = \frac{p^a}{(a-1)!}[-\ln F(w)]^{a-1}[F(w)]^{p-1} f(w), \quad a \geq 1, \tag{1}$$

and

$$f_{V_b^{(p)}, V_a^{(p)}}(w,v) = \frac{p^a}{(b-1)!(a-b-1)!}[-\ln F(w)]^{b-1}\frac{f(w)}{F(w)}$$
$$\times[\ln F(w) - \ln F(v)]^{a-b-1}[F(v)]^{p-1} f(v),\ v < w, 1 \leq b < a,\ a \geq 2, \tag{2}$$

respectively by Pawlas and Szynal (1998).

Let (W_i, V_i), $i = 1,2,...,$ be a random bivariate sample, with a common continuous $F_{W,V}(w,v)$ $= P(W \leq w, V \leq v)$. When the investigator is just interested in studying the sequence of k-records of the first component W, the second component associated with the k-RV of the first one is termed as the concomitant of that k-RV.

The pdf of the concomitants of $V_{[a,\,p]}^{(l)}$ (the a- th lower concomitant of $W_{[a,\,p]}^{(l)}$) is given as

$$g_{[a,\,p]}^{(l)}(v) = \int f_{V|W}(v \mid w) f_{V_a^{(p)}}(w)dw, \tag{3}$$

The joint pdf of concomitants of $V_{[a,\,p]}^{(l)}$ and $V_{[b,\,p]}^{(l)}$ $(a < b)$ is

$$g^{l}_{[a,b,p]} = \int_{0}^{\infty}\int_{0}^{w_1} f_{V|W}(v_1 \mid w_1) f_{V|W}(v_2 \mid w_2) f_{V_b^{(p)},V_a^{(p)}}(w_1, w_2) dw_1 dw_2 \tag{4}$$

Where $f_{V|W}(v \mid w)$ is the conditional pdf of V given W.

An excellent review on concomitants of order statistics is given by Ahsanullah (2009), Mohsin *et., al.* (2009) studied the concomitants of lower records for bivariate pseudo inverse Rayleigh distribution, Ahsanullah (2010) studied the concomitants of upper record statistics for bivariate Pseudo–Weibull distribution, Shahbaz *et., al.* (2010) studied on distribution of bivariate concomitants of records, Tahmasebi and Jafari (2015) studied the concomitants of order statistics and record values from Morgenstern type bivariate-generalized exponential distribution, Paul and Thomas (2017), Barakat, *et., al.* (2018) studied the concomitants of order statistics and record values from Bairamov-Kotz-Bekci FGM bivariate-generalized exponential distribution, Barakat, *et., al.* (2021) studied the concomitants of order statistics and record values from Iterated FGM type bivariate-generalized exponential distribution.

2. Bivariate Pseudo Powered Inverse Rayleigh Distribution

Pseudo distributions are a very recent addition to probability distributions. Fillus and Fillus (2001 and 2006) presented the Pseudo-Normal and Pseudo-Gamma distributions as linear mixtures of normal and gamma random variables. In this section we characterise the Bivariate Pseudo- powered inverse Rayleigh (*BPPIR*) distribution in accordance with the framework established by Shahbaz and Ahmad (2009) as follows:

Let random variable W has a powered inverse Rayleigh (*PIR*) distribution with parameter θ and β. The density function of W is:

$$f(w;\theta,\beta) = \frac{2\theta\beta}{w^{2\beta+1}} e^{-\frac{\theta}{w^{2\beta}}}, \; 0 < w < \infty, \; \theta, \beta > 0. \tag{5}$$

The marginal pdf of W is

$$F(w;\theta,\beta) = e^{-\frac{\theta}{w^{2\beta}}}, \; 0 < w < \infty, \; \theta, \beta > 0. \tag{6}$$

Further, let another random variable V has *PIR* distribution with parameter $\phi(w)$ and β. The density function of v is:

$$f(v;\phi(w),\beta) = \frac{2\phi(w)\beta}{v^{2\beta+1}} e^{-\frac{\phi(w)}{v^{2\beta}}}, \; 0 < w < \infty, 0 < v < \infty, \; \phi(w), \beta > 0. \tag{7}$$

The *BPPIR* distribution is defined as the compound distribution of (5) and (6). The density function of the *BPPIR* distribution is given as:

$$f(w,v) = \frac{4\theta\beta^2}{w^{2\beta+1}} \frac{\phi(w)}{v^{2\beta+1}} \exp\left[-\left\{\frac{\phi(w)}{v^{2\beta}} + \frac{\theta}{w^{2\beta}}\right\}\right]; \quad \phi(w) > 0, \theta > 0, \beta > 0, v > 0, w > 0. \tag{8}$$

Several distributions, can be obtained from (7) depending upon various choices of $\phi(w)$ and we have used $\phi(w) = w^{-2\beta}$ to obtained the following *BPPIR* distribution.

$$f(w,v) = \frac{4\theta\beta^2}{w^{4\beta+1}}\frac{1}{v^{2\beta+1}} \exp\left[-\frac{1}{w^{2\beta}}\left\{\theta + \frac{1}{v^{2\beta}}\right\}\right]; \quad \theta > 0, \beta > 0, v > 0, w > 0. \tag{9}$$

The conditional pdf of *V* given *W* for (8) is

$$f(v \mid w) = \frac{2\beta}{w^{2\beta}}\frac{1}{v^{2\beta+1}} \exp\left[-\frac{1}{w^{2\beta}}\frac{1}{v^{2\beta}}\right]; \quad \theta > 0, \beta > 0, v > 0, w > 0. \tag{10}$$

Mohsin *et. al* (2009) studied distribution (9) by using $\beta = 1$ for which (9) is simply equal to the product to two marginal densities.

In following section, the distribution of concomitant of *LRV* has been derived for (9).

3. Distribution of Concomitants of p-th *LRV* and its Properties

For the *BPPIR* distribution as given by (8) and (9), the pdf of $g_{[a,\,p]}^{(l)}(v)$ in view of (10), (5), (6), (1) and (3) is given as

$$g_{[a,\,p]}^{(l)}(v) = \frac{(2\beta)^2}{\Gamma(a)}\frac{(p\theta)^a}{v^{2\beta+1}}\int_0^\infty \frac{1}{w^{4\beta+1}}\left(\frac{1}{w^{2\beta}}\right)^{a-1} \exp\left[-\frac{1}{w^{2\beta}}\left(p\theta + \frac{1}{v^{2\beta}}\right)\right]dw, \tag{11}$$

Let, $t = w^{-2\beta}$, then the R.H.S. of (11) reduces to

$$g_{[a,\,p]}^{(l)}(v) = \frac{(2\beta)}{\Gamma(a)}\frac{(p\theta)^a}{v^{2\beta+1}}\int_0^\infty t^{(a+1)-1} \exp\left[-t\left(p\theta + \frac{1}{v^{2\beta}}\right)\right]dt, \tag{12}$$

Since,

$$\int_0^\infty x^{\alpha-1}e^{-\sigma x}dx = \frac{\Gamma(\alpha)}{\sigma^\alpha}, \quad [\operatorname{Re}\sigma > 0], \tag{13}$$

Thus applying (13) in (12), we have

$$g_{[a,\,p]}^{(l)}(v) = 2a\beta\,\frac{(p\theta)^a}{v^{2\beta+1}}\left(\frac{v^{2\beta}}{1+p\theta v^{2\beta}}\right)^{a+1}, \; v > 0. \tag{14}$$

The *m–th* moment of the distribution given in (14) is obtained as:

$$E(V_{[a,\,p]}^{(m)}) = 2a\beta\,\frac{(p\theta)^a}{v^{2\beta+1}}\int_0^\infty \frac{v^{m+2a\beta-1}}{(1+p\theta v^{2\beta})^{a+1}}dv, \tag{15}$$

Let, $u = v^{-2\beta}$, then the R.H.S. of (15) reduces to

$$E(V_{[a,\,p]}^{(m)}) = a(p\theta)^a\int_0^\infty \frac{u^{(\frac{m}{2\beta}+a)-1}}{(1+p\theta u)^{a+1}}du, \tag{16}$$

Noting that [Gradshteyn & Ryzhik, (2007)]

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+\gamma x)^{\nu}} dx = \frac{1}{\gamma^{\alpha}} B(\alpha, \nu-\alpha), \quad [|\arg \gamma| < \pi, \quad \operatorname{Re}\nu > \operatorname{Re}\alpha > 0], \tag{17}$$

Thus applying (17) in (16), we get after simplification

$$E(V_{[a,\, p]}^{(m)}) = \frac{1}{(p\theta)^{m/2\beta}} \frac{\Gamma(m/2\beta + a)\Gamma(1-m/2\beta)}{\Gamma(a)}, \ m < 2. \tag{18}$$

Set $\beta = 1$ and $p = 1$ in (18), we get moments of concomitants of RV for *BPPIR* distribution as

$$E(V_{[a,\, 1]}^{(m)}) = \frac{1}{(\theta)^{m/2}} \frac{\Gamma(m/2 + a)\Gamma(1-m/2)}{\Gamma(a)}, \ m < 2.$$

as obtained by (Mohsin *et. al*, 2009).

The mean of the concomitants of lower record statistics for *BPPIR* distribution can be obtained by using (18). Table 1 shows numerical values of mean of the concomitants of lower record statistics for *BPPIR* distribution for different values of β, θ and p.

Table 1. Means of concomitant observations at specified *p-th LRV* for *BPPIR* distribution

a	$\beta = 1,\ \theta = 1$ $p = 1$	$\beta = 2,\ \theta = 2$ $p = 2$	$\beta = 3,\ \theta = 3$ $p = 3$	$\beta = 4,\ \theta = 4$ $p = 4$
1	1.570796	0.7853982	0.7260862	0.7256133
2	2.356194	0.9817477	0.8471006	0.8163149
3	2.945243	1.104466	0.9176923	0.8673346
4	3.436117	1.196505	0.9686752	0.9034736
5	3.865632	1.271287	1.009037	0.9317071
6	4.252195	1.334851	1.042671	0.9549998
7	4.606544	1.39047	1.071634	0.9748956
8	4.935583	1.440129	1.097149	0.9923045
9	5.244057	1.485133	1.120007	1.007809
10	5.535394	1.526387	1.140748	1.021807

4. Joint Probability Density Function and Product Moments of Two Concomitants $V_{[a,\, p]}^{(m)}$ and $V_{[b,\, p]}^{(n)}$

For the *BPPIR* distribution as given in (9), the joint *pdf* of $V_{[a,\, p]}^{(m)}$ and $V_{[b,\, p]}^{(n)}$ in view of (2), (4), (5), (6) and (10) is given as

$$g_{[a,b,\, p]}^{(l)}(v_1, v_2) = \frac{(2\beta)^4}{\Gamma(b)\Gamma(a-b)} \frac{(p\theta)^a}{v_1^{2\beta+1} v_2^{2\beta+1}} \int_0^\infty \frac{1}{w_1^{4b\beta+2\beta+1}} \exp\left(-\frac{1}{w_1^{2\beta}} \frac{1}{v_1^{2\beta}}\right) I(w_2, v_1) dw_1, \tag{19}$$

where

$$I(w_2,v_1)=\int_0^{w_1}\frac{1}{w_2^{4\beta+1}}\left(\frac{1}{w_2^{2\beta}}-\frac{1}{w_1^{2\beta}}\right)^{a-b-1}\exp\left[-\frac{1}{w_2^{2\beta}}\left(p\theta+\frac{1}{v_2^{2\beta}}\right)\right]dw_2. \tag{20}$$

Let $t_1=\frac{1}{w_2^{2\beta}}-\frac{1}{w_1^{2\beta}}$, then the R.H.S. of (20) reduces to

$$I(w_2,v_1)=\frac{\exp\left[-\frac{1}{w_1^{2\beta}}\left(p\theta+\frac{1}{v_2^{2\beta}}\right)\right]}{2\beta}\int_0^{w_1}t^{a-b-1}\left(t+\frac{1}{w_1^{2\beta}}\right)\exp\left[-t\left(p\theta+\frac{1}{v_2^{2\beta}}\right)\right]dt_1. \tag{21}$$

Thus applying (13) in (21), we have

$$I(w_2,v_1)=\frac{\Gamma(a-b)}{2\beta w_1^{2\beta}}\frac{\left(p\theta+\frac{1}{v_2^{2\beta}}\right)^{b-a}}{(1+p\theta v_2^{2\beta})}[1+v_2^{2\beta}\{p\theta+(a-b)w_1^{2\beta}\}]\exp\left[-\frac{1}{w_1^{2\beta}}\left(p\theta+\frac{1}{v_2^{2\beta}}\right)\right]. \tag{22}$$

Now putting the value of $I(w_2,v_1)$ from (22) in (19), we obtain

$$g_{[a,b,\,p]}^{(l)}(v_1,v_2)=\frac{(2\beta)^3}{\Gamma(b)}\frac{(p\theta)^a}{v_1^{2\beta+1}v_2^{2\beta+1}}\frac{\left(p\theta+\frac{1}{v_2^{2\beta}}\right)^{b-a}}{(1+p\theta v_2^{2\beta})}$$

$$\times\int_0^{\infty}\frac{1}{w_1^{2b\beta+4\beta+1}}[1+v_2^{2\beta}\{p\theta+(a-b)w_1^{2\beta}\}]\exp\left[-\frac{1}{w_1^{2\beta}}\left(p\theta+\frac{1}{v_1^{2\beta}}+\frac{1}{v_2^{2\beta}}\right)\right]dw_1. \tag{23}$$

Setting $t_2=w_1^{-2\beta}$ in (23), then the R.H.S. of (23) reduces to

$$g_{[a,b,\,p]}^{(l)}(v_1,v_2)=\frac{(2\beta)^2}{\Gamma(b)}\frac{(p\theta)^a}{v_1^{2\beta+1}v_2^{2\beta+1}}\frac{\left(p\theta+\frac{1}{v_2^{2\beta}}\right)^{b-a}}{(1+p\theta v_2^{2\beta})}$$

$$\times\int_0^{\infty}t_2^{b+1}[1+v_2^{2\beta}\{p\theta+(a-b)t_2^{-1}\}]\exp\left[-t_2\left(p\theta+\frac{1}{v_1^{2\beta}}+\frac{1}{v_2^{2\beta}}\right)\right]dt_2. \tag{24}$$

Therefore, after using (13) in (24), we finally get

$$g_{[a,b,\,p]}^{(l)}(v_1,v_2)=4b\beta^2(p\theta)^a v_1^{2b\beta-1}v_2^{2a\beta+2\beta+1}\ (1+p\theta v_2^{2\beta})^{b-a-1}$$

$$\times\frac{\left[(1+a)(1+p\theta v_2^{2\beta})v_1^{2\beta}+(a-b)v_2^{2\beta}\right]}{\left[v_1^{2\beta}+v_2^{2\beta}+p\theta v_1^{2\beta}v_2^{2\beta}\right]^{(b+2)}}. \tag{25}$$

For *BPPIR* distribution, the Product Moments of Two Concomitants $V_{[a,\,p]}^{(m)}$ and $V_{[a,\,p]}^{(m)}$ is given as

$$E\left(V_{[a,\,p]}^{(m)}V_{[b,\,p]}^{(n)}\right)=\int_0^{\infty}\int_0^{\infty}v_1^m v_2^n\ g_{[a,b,\,p]}^{(l)}(v_1,v_2)dv_1dv_2. \tag{26}$$

In view of (25) and (26), we have

$$E\left(V_{[a,\,p]}^{(m)}V_{[b,\,p]}^{(n)}\right)=4b\beta^{2}(p\theta)^{a}\int_{0}^{\infty}\int_{0}^{\infty}v_1^{m+2b\beta-1}v_2^{n+2a\beta+2\beta+1}(1+p\theta v_2^{2\beta})^{b-a-1}$$

$$\times\frac{\left[(1+a)(1+p\theta v_2^{2\beta})v_1^{2\beta}+(a-b)v_2^{2\beta}\right]}{\left[v_1^{2\beta}+v_2^{2\beta}+p\theta v_1^{2\beta}v_2^{2\beta}\right]^{(b+2)}}dv_1dv_2. \tag{27}$$

$$E\left(V_{[a,\,p]}^{(m)}V_{[b,\,p]}^{(n)}\right)=4b\beta^{2}(p\theta)^{a}\int_{0}^{\infty}v_2^{n+2a\beta+2\beta+1}(1+p\theta v_2^{2\beta})^{b-a-1}I(v_1)dv_2. \tag{28}$$

Where,

$$I(v_1)=\int_{0}^{\infty}v_1^{m+2b\beta-1}\frac{\left[(1+a)(1+p\theta v_2^{2\beta})v_1^{2\beta}+(a-b)v_2^{2\beta}\right]}{\left[v_1^{2\beta}+v_2^{2\beta}+p\theta v_1^{2\beta}v_2^{2\beta}\right]^{(b+2)}}dv_1. \tag{29}$$

Now letting $q_1=v_1^{2\beta}$ in (29), we have

$$I(v_1)=\frac{1}{2\beta}\frac{1}{(v_2^{2\beta})^{(b+2)}}\int_{0}^{\infty}q_1^{m/2+b-1}\frac{\left[(1+a)(1+p\theta v_2^{2\beta})q_1+(a-b)v_2^{2\beta}\right]}{\left[1+\left(p\theta+\frac{1}{v_2^{2\beta}}\right)q_1\right]^{(b+2)}}dq_1. \tag{30}$$

Now applying (17) in (30), we get

$$I(v_1)=\frac{(m/2\beta+a)}{2\beta}\frac{\Gamma(m/2\beta+b)\Gamma(1-m/2\beta)}{\Gamma(b+1)}\frac{(v_2^{2\beta})^{m/2\beta\,-1}}{(1+p\theta v_2^{2\beta})^{m/2\beta+b}}. \tag{31}$$

Putting the values of $I(v_1)$ in (28), we get

$$E\left(V_{[a,\,p]}^{(m)}V_{[b,\,p]}^{(n)}\right)=2b\beta(p\theta)^{a}\frac{(m/2\beta+a)}{2\beta}\frac{\Gamma(m/2\beta+b)\Gamma(1-m/2\beta)}{\Gamma(b+1)}$$

$$\times\int_{0}^{\infty}v_2^{n+2a\beta+2\beta+1}\frac{(v_2^{2\beta})^{m/2\beta-1}}{(1+p\theta v_2^{2\beta})^{b-a-1}}dv_2. \tag{32}$$

Further setting $q_2=v_2^{2\beta}$ in (32), it becomes

$$E\left(V_{[a,\,p]}^{(m)}V_{[b,\,p]}^{(n)}\right)=b(p\theta)^{a}\frac{(m/2\beta+a)}{2\beta}\frac{\Gamma(m/2\beta+b)\Gamma(1-m/2\beta)}{\Gamma(b+1)}$$

$$\times\int_{0}^{\infty}\frac{q_2^{(n+m)/2+a-1}}{(1+p\theta q_2)^{m/2+a+1}}dq_2. \tag{33}$$

Finally applying (17) in (33), we get

$$E\left(V_{[a,\,p]}^{(m)}V_{[b,\,p]}^{(n)}\right)=\frac{\Gamma(1-m/2\beta)\Gamma(1-n/2\beta)\Gamma(m/2\beta+b)\Gamma(m+n+2a)/2\beta}{(p\theta)^{(n+m)/2\beta}\quad\Gamma(b)},\ n<2,\ m<2. \tag{34}$$

Set $\beta=1$ and $p=1$ in (34), we get the product moment of concomitants of RV from $BPPIR$ distribution as

$$E\left(V^{(m)}_{[a,\,1]}\,V^{(n)}_{[b,\,1]}\right)=\frac{\Gamma(1-m/2)\Gamma(1-n/2)\Gamma(m/2+b)\Gamma(m+n+2a)/2}{(\theta)^{(n+m)/2}\quad\Gamma(b)},\ n<2,\ m<2.$$

as obtained by (Mohsin *et. al*, 2009).

Table 2. Covariance between concomitant variables of record values under *BPPIR* distribution

a	b	$\theta=1,\ \beta=1$ $p=1$	$\theta=2,\ \beta=2$ $p=2$	$\theta=3,\ \beta=3$ $p=3$	$\theta=4,\ \beta=4$ $p=4$
1	1	0.1787143	0.05198035	0.2146661	0.4262263
	1	1.05347	-0.1370497	-0.04145286	0.0851498
2	2	1.580204	-0.1713121	-0.04836167	0.09579352
	1	6.814624	-0.1525178	-0.140739	-0.06448303
3	2	10.22194	-0.1906472	-0.1641955	-0.07254341
	3	12.77742	-0.2144781	-0.1778785	-0.07707737
	1	34.65393	-0.02860502	-0.168617	-0.136572
4	2	51.9809	-0.03575628	-0.1967198	-0.1536435
	3	64.97612	-0.04022581	-0.2131132	-0.1632462
	4	75.80548	-0.04357796	-0.2249528	-0.1700481

5. Application

Finding the minimum variance linear unbiased estimates (MVLUE) of the location and scale parameters is an intriguing application of this work. Although the Llyod (1952) technique can be applied, numerical methods were utilized to obtain these estimates because it was challenging to obtain the inverse of the variance-covariance matrix in closed form.

Assume the random variables have the location parameter μ and scale parameter σ . Utilizing Lloyd's method, the MVLUE of θ as $\hat{\theta}=(M'V_*^{-1}v)^{-1}(M'V_*^{-1}M)^{-1}$ where $V_*=(V_{*_{i,j}})$ is the variance of the $i-th$ and $j-th$ concomitants, V_*^{-1} is the inverse of the matrix V_*, w' is the observed value of the vector $V'=V_{[1,p]},V_{[2,p]},\ldots,V_{[a,p]}$, $\mu=0$ and $\sigma=1$ M and θ are defined as:

$$M=\begin{bmatrix} 1 & 1 & 1 & \cdots\ 1 \\ \mu_{[1,p]} & \mu_{[2,p]} & \mu_{[3,p]} & \cdots\ \mu_{[a,p]} \end{bmatrix},\ \theta'=[\mu\quad\sigma].$$

Table 3. Estimated MVLUE coefficients of parameters μ and σ based on $p-th$ *LRV* in *BPPIR* model

$(\theta,\ \beta, p)$	Estimate	Coefficients			
(1,1,1)	$\hat{\mu}$	2.886238	-1.739833	-0.138116	-0.008289
	$\hat{\sigma}$	-4.758239	8.747066	-3.203087	-0.785739
(2,2,2)	$\hat{\mu}$	5.582690	-5.353715	0.513147	0.257878
	$\hat{\sigma}$	-4.788459	4.695073	-0.398794	0.492179
(3,3,3)	$\hat{\mu}$	-23.75326	73.58569	-42.89371	-5.938716
	$\hat{\sigma}$	30.23725	-91.24636	53.39990	7.609213
(4,4,4)	$\hat{\mu}$	5.159918	6.706993	-16.49483	5.627918
	$\hat{\sigma}$	-5.813592	-8.553672	21.47709	-7.109824

6. Survival Analysis

In this section we have conducted the survival analysis of the distribution of the concomitant of lower record statistics for Bivariate Pseudo- powered inverse Rayleigh distribution. We recall that the survival function and hazard functions for a probability distribution are given as:

$$S(t)=1-F(t) \text{ and } \quad h(t)=\frac{f(t)}{S(t)}$$

These functions are useful for the survival analysis of any parent probability distribution. In this section we have obtained these functions for the distribution (14). For this, we first see that the distribution function of (14) is given as:

$$F(t)=\int_0^t g_{[a,\ p]}^{(l)}(v)dv, \tag{35}$$

In view of (14) and (35), we get

$$F(t)=2a\beta(p\theta)^a\int_0^t \frac{1}{v^{2\beta+1}}\left(\frac{v^{2\beta}}{1+p\theta v^{2\beta}}\right)^{a+1} dv, \tag{36}$$

Setting $u=v^{-2\beta}$ in (36), then the R.H.S. of (36) reduces to

$$F(t)=a(p\theta)^a\int_{t^{-2\beta}}^{\infty} (u+p\theta)^{-(a+1)}du,$$

After simplification, we get

$$F(t)=(p\theta)^a\left(\frac{t^{2\beta}}{1+p\theta t^{2\beta}}\right)^a, 0<t<\infty,\ \theta,\beta>0. \tag{37}$$

Using (37), the survival function is:

$$S(t)=1-(p\theta)^a\left(\frac{t^{2\beta}}{1+p\theta t^{2\beta}}\right)^a, \tag{38}$$

Further, by using (14) and (4.2), the hazard function is:

$$h(t)=\frac{\dfrac{2a\beta(p\theta)^{a}}{t^{2\beta+1}}\left(\dfrac{t^{2\beta}}{1+p\theta t^{2\beta}}\right)^{a+1}}{\left[1-(p\theta)^{a}\left(\dfrac{t^{2\beta}}{1+p\theta t^{2\beta}}\right)^{a}\right]}. \tag{39}$$

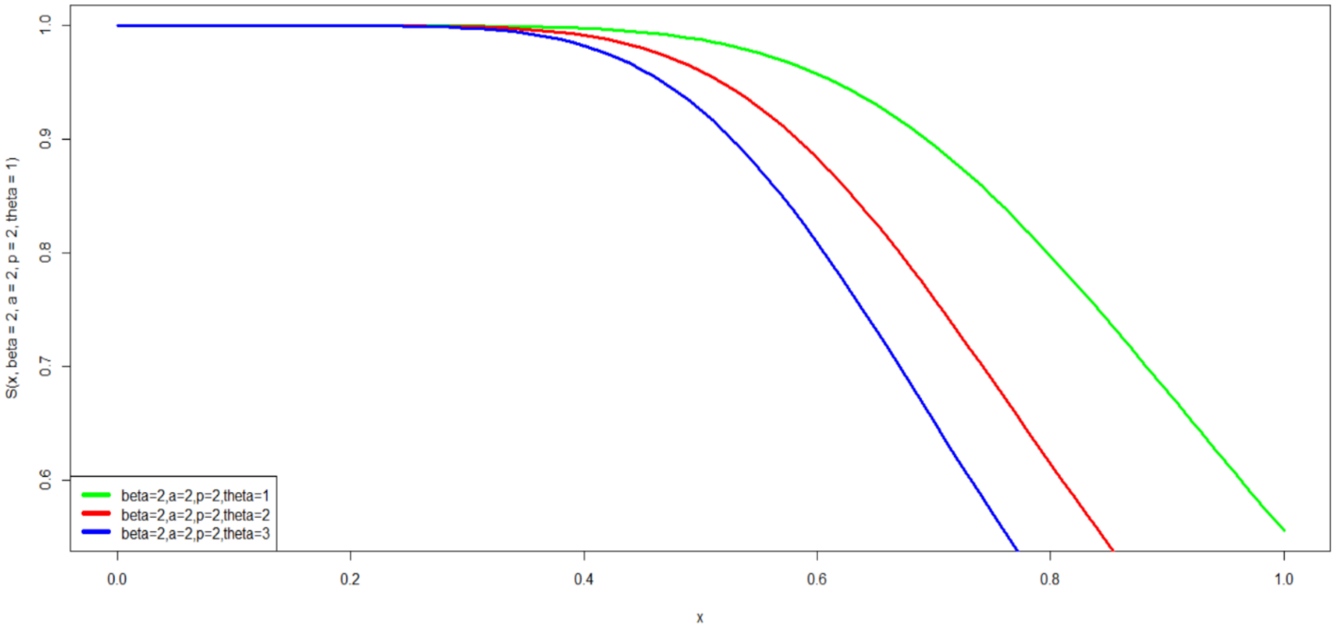

Figure 1. Survival function at different values of θ holding t, p,β and a at a fixed level.

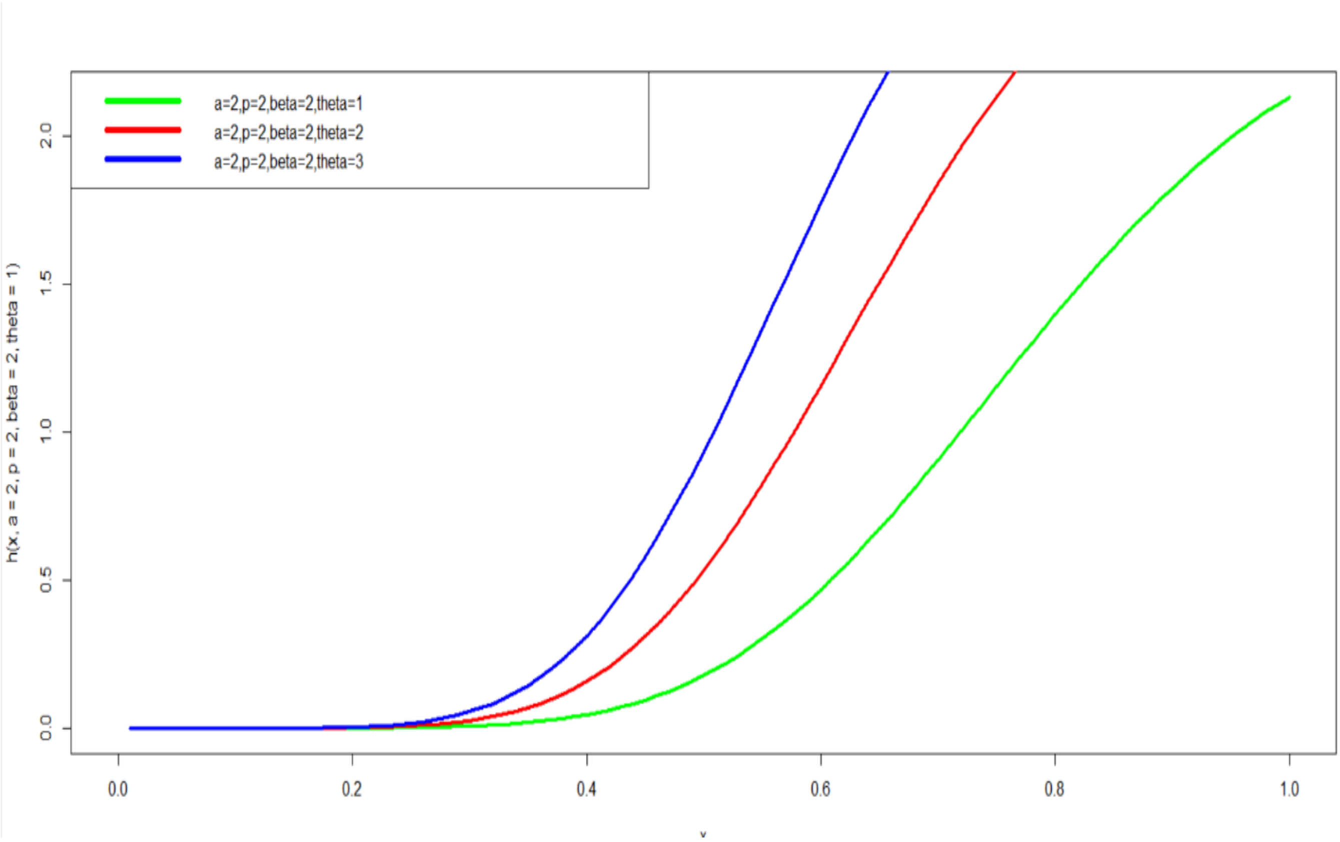

Figure 2. Hazard rate function at different values of θ holding t, p,β and a at a fixed level.

We have obtained the numeric values of the survival function and hazard function for $a = 8$ and for various values of p, θ, and β. These values are obtained for $a = 1(1)8$ and for $t = 0.1(1.0)0.1$ and are given in Table 4 and Table 5.

Table 4. Survival probabilities of concomitant variables associated with *LR* Statistics for the *BPPIR* model

$p = 2,\ \beta = 2,\ \theta = 1$

	a							
t	1	2	3	4	5	6	7	8
0.1	0.999800	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.2	0.996810	0.999989	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.3	0.984058	0.999745	0.999995	0.999999	1.000000	1.000000	1.000000	1.000000
0.4	0.951293	0.997627	0.999884	0.999994	0.999999	1.000000	1.000000	1.000000
0.5	0.888888	0.987654	0.998628	0.999847	0.999983	0.999998	0.999999	1.000000
0.6	0.794155	0.957627	0.991277	0.998204	0.999630	0.999923	0.999984	0.999996
0.7	0.675584	0.894754	0.965856	0.988923	0.996406	0.998834	0.999621	0.999877
0.8	0.549692	0.797222	0.908687	0.958881	0.981484	0.991662	0.996245	0.998309
0.9	0.432488	0.677930	0.817222	0.896271	0.941132	0.966592	0.981040	0.989240
1.0	0.333333	0.555555	0.703703	0.802469	0.868312	0.912208	0.941472	0.960981

$p = 2,\ \beta = 2,\ \theta = 2$

	a							
t	1	2	3	4	5	6	7	8
0.1	0.999600	0.999999	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.2	0.993640	0.999959	0.999999	1.000000	1.000000	1.000000	1.000000	1.000000
0.3	0.968616	0.999015	0.999969	0.999999	1.000000	1.000000	1.000000	1.000000
0.4	0.907111	0.991371	0.999198	0.999925	0.999993	0.999999	0.999999	1.000000
0.5	0.800000	0.960000	0.992000	0.998400	0.999680	0.999936	0.999987	0.999997
0.6	0.658588	0.883437	0.960204	0.986413	0.995361	0.998416	0.999459	0.999815
0.7	0.510100	0.759998	0.882423	0.942399	0.971781	0.986175	0.993227	0.996682
0.8	0.379017	0.614380	0.760537	0.851297	0.907658	0.942657	0.964391	0.977887
0.9	0.275907	0.475690	0.620351	0.725099	0.800946	0.855867	0.895634	0.924429
1.0	0.200000	0.360000	0.488000	0.590400	0.672320	0.737856	0.790284	0.832227

In the **Table 4** contains the values of survival function (38). Looking at these tables we can see that the survival probability of the concomitant decreases with increase in the value of θ holding t, p, β and a at a fixed level. Further, from the same table we can see that; for fixed p, θ, β and t; the survival probability increase with increase in a. Also the table 4 shows that; for fixed p, θ, β and a; the survival probability decreases with increase in t.

Table 5. Hazard function dynamics for concomitants of *LR* statistics in *BPPIR* distribution

$p=1, \quad \beta=1, \ \theta=1$

	a							
t	1	2	3	4	5	6	7	8
0.1	0.198019	0.003882	5.76e-05	7.61e-07	9.42e-09	1.11e-10	1.29e-12	1.46e-14
0.2	0.384615	0.028490	0.001641	8.41e-05	4.04e-06	1.86e-07	8.38e-09	3.68e-10
0.3	0.550458	0.083968	0.010334	0.001137	0.000117	1.16e-05	1.12e-06	1.05e-07
0.4	0.689655	0.167189	0.034022	0.006243	0.001076	0.000178	2.86e-05	4.51e-06
0.5	0.800000	0.266666	0.077419	0.020513	0.005122	0.001228	0.000286	6.55e-05
0.6	0.882352	0.369357	0.138957	0.048372	0.015947	0.005060	0.001562	0.000472
0.7	0.939597	0.465053	0.212140	0.090773	0.037020	0.014571	0.005585	0.002098
0.8	0.975609	0.547710	0.288958	0.144774	0.069613	0.032419	0.014728	0.006563
0.9	0.994475	0.614904	0.362600	0.205197	0.112195	0.059648	0.031004	0.015825
1.0	1.000000	0.666667	0.428571	0.266667	0.161290	0.095238	0.055118	0.031372

$p=1, \ \beta=1, \ \theta=2$

	a							
t	1	2	3	4	5	6	7	8
0.1	0.392156	0.015080	0.000443	1.15e-05	2.84e-07	6.68e-09	1.52e-10	3.42e-12
0.2	0.740740	0.102171	0.011294	0.001115	0.000103	9.17e-06	7.93e-07	6.71e-08
0.3	1.016949	0.269192	0.060375	0.012242	0.002333	0.000427	7.60e-05	1.32e-05
0.4	1.212121	0.473022	0.164239	0.052512	0.015871	0.004614	0.001304	0.000361
0.5	1.333333	0.666666	0.307692	0.133333	0.055096	0.021978	0.008539	0.003252
0.6	1.395349	0.823484	0.460223	0.245567	0.126170	0.062903	0.030623	0.014631
0.7	1.414141	0.936390	0.597317	0.368509	0.220870	0.129186	0.074039	0.041726
0.8	1.403509	1.009265	0.707164	0.483729	0.323796	0.212628	0.137318	0.087418
0.9	1.374046	1.049978	0.787736	0.580803	0.421370	0.301227	0.212496	0.148137
1.0	1.333333	1.066667	0.842105	0.656410	0.505529	0.384962	0.290108	0.216547

In the **Table 5** contains the values of hazard function (39). Looking at these tables we can see that the hazard rate of the concomitant decreases with increase in the value of θ holding t, p, β and a at a fixed level. Further, from the table we can see that; for fixed p, θ, β and t; the hazard rate function decreases with increase in a. Also the table shows that; for fixed p, θ, β and a; the function increase with increase in t.

7. Expression for Reliability

In the area of stress-strength models, the component fails when stress (W) exceeds the strength (V) otherwise the component works adequately. The algebraic form for $R = P_r(W < V)$ has been worked out for the vast majority of distributions when W and V are independent random variables having the same univariate distribution. In this section we derive the expression for R when W and V are correlated with *BPPIR* distribution (9).

The reliability coefficient is defined as:

$$R = P_r(W < V) = \int_0^{\infty} \int_w^{\infty} f(w,v)dwdv. \tag{40}$$

In view of (9) and (40), we get

$$R = (2\beta)^2 \theta \int_0^{\infty} \frac{1}{w^{4\beta+1}} \exp\left(-\frac{\theta}{w^{2\beta}}\right) I(w)dw, \tag{41}$$

where

$$I(w) = \int_w^{\infty} \frac{1}{v^{2\beta+1}} \exp\left(-\frac{1}{w^{2\beta}} \frac{1}{v^{2\beta}}\right)dv, \tag{42}$$

Now letting $z = w^{-2\beta} v^{-2\beta}$ in (42), we have

$$I(w) = \frac{w^{2\beta}}{2\beta}\left[1 - \exp\left(-\frac{1}{w^{4\beta}}\right)\right], \tag{43}$$

Putting the values of $I(w)$ in (41), we get

$$R = 2\beta\theta \int_0^{\infty} \frac{1}{w^{2\beta+1}} \exp\left(-\frac{\theta}{w^{2\beta}}\right)\left[1 - \exp\left(-\frac{1}{w^{4\beta}}\right)\right]dw, \tag{44}$$

Further setting $u = w^{-2\beta}$ in (44), it becomes

$$R = 1 - \theta \int_0^{\infty} \exp(-u^2 - \theta u)du, \tag{45}$$

Noting that [Gradshteyn and Ryzhik, (2007)]

$$\int_0^\infty \exp\left(-\frac{x^2}{4\beta}-\alpha x\right)dx=\sqrt{\pi\beta}\exp(\beta\alpha^2)[1-\Phi(\alpha\sqrt{\beta})],\ \operatorname{Re}\beta>0. \tag{46}$$

Finally applying (46) in (45), we get

$$R=1-\sqrt{\frac{\pi}{4}}\exp(\theta^2/4)[1-\Phi(\theta/2)],\ \operatorname{Re}\beta>0. \tag{47}$$

Conflicts of Interest

The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

Author Contributions

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